Classification of two-dimensional binary cellular automata with respect to surjectivity

Henryk Fukś and Andrew Skelton Department of Mathematics Brock University St. Catharines, ON, Canada

Abstract—While the surjectivity of the global map in two-dimensional cellular automata (2D CA) is undecidable in general, in specific cases one can often decide if the rule is surjective or not. We attempt to classify as many 2D CA as possible by using a sequence of tests based on the balance theorem, injectivity of the restriction to finite configurations, as well as permutivity. We introduce the notion of slice permutivity which is shown to imply surjectivity in 2D CA. The tests are applied to 2D binary CA with neighbourhoods consisting of up to five sites, considering all possible contiguous shapes of the neighbourhood. We find that if the size of the neighbourhood is less than five, complete classification of all rules is possible. Among 5-site rules, those with von Neuman neighbourhoods as well as neighbourhoods corresponding to T, V, and Z pentominos can also be completely classified.

Keywords: cellular automata, surjective, permutive, classification, neighbourhood

1. Introduction

In the theory of cellular automata (CA), the surjectivity of the global map is one of the most extensively studied properties of CA. It is only natural to ask, therefore, what are the examples of surjective CA?

In the case of one-dimensional CA, such examples are easy to construct because there exists the wellknown Amoroso-Patt algorithm for determining if a given elementary cellular automaton is surjective [1]. Using this algorithm it can be shown that among the 88 minimal elementary CA rules, the only surjective rules have Wolfram code numbers 15, 30, 45, 51, 60, 90, 105, 106, 150, 154, 170 and 204.

In two dimensions, however, the situation is much different. It has been shown that the question of surjectivity of two-dimensional cellular automata is undecidable [4], which means that it is impossible to construct a single algorithm which would always decide if an arbitrary rule is surjective or not. This, of course, does not exclude a possibility that for specific classes of 2D rules surjectivity can still be decidable – it is known, for example, that rules which are permutive with respect to the corners of the Moore neighbourhood are surjective [2].

In this paper, we attempt to classify 2D rules with respect to surjectivity using two known properties equivalent to surjectivity, namely the balance theorem and the injectivity of restrictions to finite configurations. Moreover, we introduce the the concept of slice-permutivity which is then shown to imply surjectivity. We show that all 2D CA with neighbourhoods of size four (or less), no matter what shape, can be classified. For five-site CA, complete classification is still possible for certain neighbourhood shapes, notably including von Neumann neighbourhood.

2. Basic Definitions

Let \mathcal{A} be a finite set of symbols, to be called a *symbol* set. We define a *two-dimensional configuration* s to be a function $s : \mathbb{Z}^2 \to \mathcal{A}$, and $\mathcal{A}^{\mathbb{Z}^2}$ to be the set of all two-dimensional configurations. For any vector $\vec{x} \in \mathbb{Z}^2$, we denote $s_{\vec{x}} \in \mathcal{A}$ to be a symbol located at position (or site) \vec{x} in configuration s. If $\mathcal{V} \subset \mathbb{Z}^2$, we define $s_{\mathcal{V}} = [s_{\vec{x}}]_{\vec{x} \in \mathcal{V}}$.

A neighbourhood \mathcal{N} is a finite subset of vectors in \mathbb{Z}^2 . A neighbourhood is said to be *contiguous* if, for any vector $\vec{x} \in \mathcal{N}$, at least one vector in the set $\{\vec{x} \pm (1,0), \vec{x} \pm (0,1)\}$ is also in \mathcal{N} . For any vector \vec{x} , the neighbourhood of \vec{x} is defined as $\mathcal{N}(\vec{x}) = \{\vec{u} + \vec{x} : \vec{u} \in \mathcal{N}\}$.

We can now define the *local mapping* of a *two*dimensional cellular automata (2D CA) to be the function $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$. The local mapping induces a global mapping $F : \mathcal{A}^{\mathbb{Z}^2} \to \mathcal{A}^{\mathbb{Z}^2}$ so that $F(s)_{\vec{x}} = f(s_{\mathcal{N}(\vec{x})})$, for all $s \in \mathcal{A}^{\mathbb{Z}^2}$ and all $\vec{x} \in \mathbb{Z}^2$.

If $\mathcal{Z} \subset \mathbb{Z}^2$ is a finite set of vectors, then we define a *block* to be an element of $\mathcal{A}^{\mathcal{Z}}$.

The neighbourhood of \mathcal{Z} is defined similarly as before, so that $\mathcal{N}(\mathcal{Z}) = \{\vec{u} + \vec{x} : \vec{u} \in \mathcal{N}, \vec{x} \in \mathcal{Z}\}$. The *block* evolution operator $\mathbf{f} : \mathcal{A}^{\mathcal{N}(\mathcal{Z})} \to \mathcal{A}^{\mathcal{Z}}$ is now defined by $\mathbf{f}(b)_{\vec{x}} = f(b_{\mathcal{N}(\vec{x})})$ for any $\vec{x} \in \mathcal{Z}$ and $b \in \mathcal{A}^{\mathcal{N}(\mathcal{Z})}$.

Given a block $b \in \mathcal{A}^{\mathbb{Z}}$, the set of preimages of b under **f** is the set of blocks $b' \in \mathcal{A}^{\mathcal{N}(\mathbb{Z})}$ such that $\mathbf{f}(b') = b$. This preimage set will be denoted $\mathbf{f}^{-1}(b)$.

Sometimes, we will need to consider the neighbourhood of a neighbourhood. We will then use the notation $\mathcal{N}^2(\mathcal{Z}) = \mathcal{N}(\mathcal{N}(\mathcal{Z}))$, and higher powers will refer to the appropriate number of neighbourhood compositions.

Let $\vec{v} \in \mathcal{N}$, and let us denote $\mathcal{M} = \mathcal{N} \setminus \vec{v}$. Let $b \in \mathcal{A}^{\mathcal{M}}$ and let us denote [x, b] to be an element of $\mathcal{A}^{\mathcal{N}}$ such that its entries with indices in \mathcal{M} are the same as corresponding entries in b, while the entry with index \vec{v} is equal to $x, x \in \mathcal{A}$. A 2D CA is *permutive with respect to* $\vec{v} \in \mathcal{N}$ if, for any choice of b, the function $x \to f([x, b])$ is one-to-one.

2.1 One-dimensional Binary Rules

Before we attempt to classify two-dimensional CA rules, let us discuss what happens in one dimension, as this will give us some important insight. As mentioned in the introduction, surjectivity in 1D is known to be decidable, and the algorithm for testing for surjectivity has been developed by Amoroso and Patt in early 70's [1]. We used this algorithm to find all surjective binary rules of a given neighbourhood size, for neighbourhood sizes ranging from 1 to 5. We also checked which of these rules are permutive. The results are given in Table 1. One can make two interesting observations from this table. First of all, the proportion of rules which are surjective decreases dramatically as the neighbourhood size increases. The second observation can be stated as the following proposition.

Proposition 2.1: Any contiguous one-dimensional binary cellular automata dependent on three or less sites is surjective if and only if it is permutive.

This means that for a binary rule to be surjective yet non-permutive a neighbourhood of at least four sites is needed. A natural question to ask, therefore, is whether this is also the case in two dimensions?

3. Permutivity and Surjectivity

As we will shortly see, permutivity alone is not enough to guarantee surjectivity in two dimensions. In [2], the authors considered 2D CA with Moore neighbourhood of radius r, where $\mathcal{N} = \{(i, j) : |i|, |j| \le r\}$. They proved that any such rule is surjective if it is permutive with respect to sites $(\pm r, \pm r)$. We will prove a similar result using an arbitrary neighbourhood and any site that can be sliced off from the neighbourhood by a straight line.

Given $m, c \in \mathbb{Q}$, we define a line $\ell = \{(x,y) : y = mx + c\}$, and the following two regions, $\ell^+ = \{(x,y) : y > mx + c\}$ and $\ell^- = \{(x,y) : y < mx + c\}$. For vertical lines $\ell = \{(x,y) : x = c\}$ we similarly define $\ell^+ = \{(x,y) : x > c\}$ and $\ell^- = \{(x,y) : x < c\}$. A site $\vec{x} \in \mathcal{N}$ can be *sliced* if there exists a set ℓ such that $\vec{x} \in \ell$ and $\mathcal{N} \setminus \vec{x} \subset \ell^+$ (or ℓ^-). A 2D CA is *slice permutive* if it is permutive with respect to a site which can be sliced.

The main result relating slice-permutivity and surjectivity can be stated as follows.

Theorem 3.1: Any two-dimensional slice permutive CA is surjective.

Before we start the proof, we will need the following classical result. Let Z_n be a square region in \mathbb{Z}^2 , defined as $Z_n = [0, 1, \dots, n-1] \times [0, 1, \dots, n-1]$, where $n \in \mathbb{N}$.

Theorem 3.2 (Balance Theorem): A 2D CA is surjective if and only if for all $n \ge 1$ and all $b, b' \in \mathcal{A}^{\mathbb{Z}_n}$, we have card $\mathbf{f}^{-1}(b) = \operatorname{card} \mathbf{f}^{-1}(b')$.

A one-dimensional version of this theorem first appeared in [3]. The proof of the two-dimensional version can be found in [5], where the authors consider a Moore neighbourhood of any radius. Since any neighbourhood can be extended to a Moore neighbourhood by adding extra sites, the Balance Theorem also holds for a CA rule with any neighbourhood shape.

Proof: [of Theorem 3.1] Consider an arbitrary block $b \in \mathcal{A}^{\mathbb{Z}_n}$. Since our CA is slice permutive, there exists a line ℓ which slices the neighbourhood at some site $\vec{u} \in \mathcal{N}$, so that all sites in $\mathcal{N} \setminus \vec{u}$ are either in ℓ^- or in ℓ^+ . Without loss of generality, we assume that all sites of $\mathcal{N} \setminus \vec{u}$ are are in ℓ^- . The proof of the other case can be obtained by replacing ℓ^{\pm} with ℓ^{\mp} . Let \vec{n} be the normal vector to ℓ oriented so that it points in the direction of ℓ^+ . Consider $\{\ell_1, \ell_2, \ldots, \ell_k\}$ to be the family of lines with the same slope as ℓ with indices increasing in the direction of \vec{n} , so that for any $\vec{x} \in \mathbb{Z}_n$ there exists ℓ_i such that $\vec{x} \in \ell_i$. We now construct the set of preimages of b, each element of which is of the form $[a_{\vec{x}}]_{\vec{x} \in \mathcal{N}(\mathbb{Z}_n)}$.

In order to illustrate this better, we will conduct the proof while simultaneously referring to an example of a

Neighbourhood Size	Total Rules	Surjective Rules			
		Permutive	Not Permutive	Total	
1	2	2	0	2	
2	16	6	0	6	
3	256	28	0	28	
4	65536	518	64	582	
5	4294967296	131502	11516	143018	

Table 1: One-dimensional binary rules

rule defined on the seven-site neighbourhood

$$\mathcal{N} = \{(-3, 1), (-3, 0), (-2, 0), (-1, 0), \\(0, 0), (0, -1), (1, -1)\}.$$

In this example, we assume that the rule is permutive with respect to the sliceable site (0,0), for which the corresponding line ℓ has slope -1/2. The neighbourhood of Z_3 is shown in Figure 3(a).

We start the construction from sites of Z_n which belong to the line l_1 . Due to slice permutivity, all sites of the preimage which are below this line can take arbitrary values, and sites which are on the line can be chosen in such a way that $b_{\vec{x}} = f(a_{\mathcal{N}(\vec{x})})$ for all $\vec{x} \in \ell_1 \cap Z_n$. This means that sites of $\mathcal{N}(Z_n)$ which belong to ℓ_1^- can take arbitrary values, and sites of Z_n which belong to ℓ_1 are uniquely determined by those arbitrary values.

In Figure 3(b) arbitrary sites are denoted by stars, and the uniquely determined site (only one in this case) is denoted by D.

We then move to ℓ_2 , and again, for every possible configuration of sites from the set $\mathcal{N}(\mathcal{Z}_n) \cap \ell_2^-$, values of sites which lie in $\mathcal{Z}_n \cap \ell_2$ will be uniquely determined. It may happen, as shown in Figure 3(b), that some sites of $\mathcal{N}(\mathcal{Z}_n) \cap \ell_2^-$ have not been labeled before. These sites can also assume arbitrary values, and thus they are marked as red stars in Figure 3(c).

We repeat the above procedure until all sites of $\mathcal{N}(\mathbb{Z}_n)$ are labeled, as shown in Figures 3(d-h). In the end, as in Figure 3(i), all sites which belong to \mathbb{Z}_n will be labeled by D, while the sites of $\mathcal{N}(\mathbb{Z}_n) \setminus \mathbb{Z}_n$ will be labeled by stars. This means that we have $\operatorname{card}(\mathcal{N}(\mathbb{Z}_n) \setminus \mathbb{Z}_n)$ sites in the preimage which can assume any values from the symbol set \mathcal{A} . Thus, for each block $b \in \mathcal{B}_r$, we have $\operatorname{card} \mathbf{f}^{-1}(b) = (\operatorname{card} \mathcal{A})^{\operatorname{card}(\mathcal{N}(\mathbb{Z}_n) \setminus \mathbb{Z}_n)}$. Since this is independent of b, by Theorem 3.2 we conclude that the CA is surjective.

4. Classification procedure

The result of the previous section gives us a method to determine surjectivity of a special class of rules, namely slice-permutive rules. In classifying rules we will also need some method for checking if a rule is not surjective – although of course a *general* algorithm of this type cannot exist. One such method involves the Balance Theorem. One can simply check if all blocks of a given size have the same number of preimages if one finds a single violation of the balance theorem, the rule is obviously not surjective. Clearly, one can perform such exhaustive check only for block of small size, and if no balance violation is found, then the test remains inconclusive. It turns out, however, that this simple method is surprisingly effective for 2D rules with small neighbourhood size.

Another test for non-surjectivity is based on the following classical result [6]. Here by a finite configuration we mean a configuration where all but a finite number of sites are in some arbitrary fixed state.

Theorem 4.1: A CA is surjective if and only if it is injective when restricted to finite configurations.

If one can find two finite configurations which are different but have the same image, then the rule is not surjective. One can perform this test exhaustively for small finite configurations. It turns out to be useful in cases where the balance theorem test is inconclusive.

We may now present a procedure for classification of 2D rules with a given neighbourhood shape. This procedure is illustrated in Figure 2. While it has been tailored for binary rules (that is, rules where card $\mathcal{A} =$ 2), it can be easily adapted for larger alphabets.

Procedure 4.1: We start with a list of all rules on a given neighbourhood and apply the following filters to remove known surjective and non-surjective rules.

- 1) First we check the simplest balance condition, that is, $\operatorname{card} \mathbf{f}^{-1}(0) = \operatorname{card} \mathbf{f}^{-1}(1)$. Rules which violate this condition are non-surjective.
- 2) We check if the rule truly depended on all given sites. If not, it means it can be treated as if it had



Fig. 1: Example: Iteration of the procedure in Theorem 3.1

smaller neighbourhood, thus we check how it was classified before (we were classifying rules with progressively increasing neighbourhood size).

- 3) We check if the rule is permutive with respect to at least one sliceable site. If it is, it is surjective.
- The following two tests are performed simultaneously owing to the fact that they required approximately the same information.
 - We check if all blocks from the set $\mathcal{A}^{\mathcal{N}(\mathcal{Z}_1)}$ have the same number of preimages. If not, the rule is non-surjective.
 - We search for violations of injectivity on finite configurations. Let *M* be the set of all sites *x* ∈ Z² for which (0,0) ∈ *N*(*x*). For each *b* ∈ *A^M* we consider a finite configuration *s* such that *s_M* = *b* and the sites which do not belong to *M* are in the state 0, as well as configuration *s'* obtained from *s* by replacing *s*_(0,0) by 1-*s*_(0,0). If *F*(*s*) = *F*(*s'*), injectivity is violated, thus the rule is not surjective.

5) We check if all blocks from the set $\mathcal{A}^{\mathcal{N}^2(\mathcal{Z}_1)}$ have the same number of preimages. If not, the rule is non-surjective.

5. Classification Results

We applied the aforementioned procedure to twodimensional rules with neighbourhoods of size 3, 4, and 5. It is not necessary to consider smaller neighbourhoods, as those are effectively one-dimensional, thus results of sec. 2.1 apply. In all cases we considered only truly two-dimensional contiguous neighbourhoods, meaning that linear neighbourhoods were excluded – again, these are included in results of sec. 2.1. Contiguous neighbourhoods have shapes known as polyominoes, that is, plane geometric figures formed by joining several equal squares edge to edge. Since rigid transformations of \mathbb{Z}^2 preserve surjectivity, we considered only shapes which are representatives of equivalence classes with respect to the group of isometries of \mathbb{Z}^2 .



Fig. 2: Surjectivity test flowchart.

5.1 Three-site neighbourhoods

There is only one contiguous three-site neighbourhoods which is truly two-dimensional: the L-shaped neighbourhood (Figure 3a), with 256 corresponding binary rules. Applying the procedure of Figure 2, we found that all these rules can be classified, and that among them 38 rules are surjective, all of which are slice permutive. The remaining rules are non-surjective.

Proposition 5.1: Any contiguous three-site binary CA is surjective if and only if it is slice permutive.

5.2 Four-site neighbourhoods

There are four contiguous two-dimensional four-site neighbourhoods, often referred to as *tetrominos* and named after their resemblance to the letters L, O, S and T (Figure 3b). There are 65536 total binary rules dependent on each of these neighbourhoods.

In the case of the L and T neighbourhoods, there are 724 slice permutive (and thus surjective) rules. In the case of the O and S neighbourhoods, there are 942 slice permutive and surjective rules. This difference seems to be due to the number of sliceable sites in



Fig. 3: Contiguous Neighbourhoods

each neighbourhood (3 and 4 respectively). In each case, following Procedure 4.1 we determined that these are the only surjective rules, thus we again obtain a complete classification of all rules truly dependent on all four sites. This can be summarized as follows.

Proposition 5.2: Any contiguous four-site twodimensional binary cellular automaton is surjective if and only if it is slice permutive.

5.3 Five-site neighbourhoods

There are eleven contiguous two-dimensional five-site neighbourhoods, often referred to as *pentominos* and named after their resemblance to the letters F, L, P, S, T, U, V, W, Z, Y and Z. There are 4294967296 binary rules associated with each neighbourhood. The results of application of Procedure 4.1 are shown in Table 2. We can observe that the classification is complete in the case of four neighbourhood shapes, T, V, X, and Z.

Proposition 5.3: Any contiguous five-site twodimensional binary cellular automaton with the neighbourhood shaped as T, V, X, or Z pentomino is surjective if and only if it is slice permutive.

Note that this includes the traditional von Neumann neighbourhood (shape X). Also note that for neighbourhood shapes for which complete classification was not possible, the vast majority of rules were nevertheless classified. In all cases the fraction of rules which were classified exceeded 99.998%.

6. Conclusions

We were able to obtain complete classification with respect to surjectivity of all 2D rules with contiguous neighbourhoods of size up to 4. In these neighbourhoods, all surjective rules are slice permutive. Among 5-site rules, those with von Neuman neighbourhoods as well as neighbourhoods corresponding to T, V, and Z pentominos can also be completely classified, and again, surjectivity and slice permutivity are equivalent for them. For the remaining pentomino shapes, only a very small fraction of rules defies classification. The worst case is the S pentomino, for which .0016% rules cannot be classified with our algorithm.

A number of open questions remain. First of all, can the remaining 5-site rules be classified? We suspect that at least some of them could be, if one performed balance and/or injectivity tests on larger blocks, although such tests would be computationally very expensive. A related question is whether there exist any truly twodimensional five-site binary rule which is surjective yet not slice permutive? Since in one dimension such rules are possible even in four-site neighbourhoods, we suspect that the answer is affirmative, although currently we cannot offer any evidence of this claim.

7. Acknowledgements

One of the authors (HF) acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada (NSERC) in the form of Discovery Grant. This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca) and Compute/Calcul Canada.

Neighbourhood	Surjective	Unknown	Neighbourhood	Surjective	Unknown
F	257106	16	V	193138	0
L	193138	1472	W	257106	3596
Р	257106	3596	X	257106	0
S	192938	67764	Y	193138	1472
Т	193138	0	Z	257106	0
U	192938	64264			

Table 2: Two-dimensional truly five-site binary rules

References

- S. Amoroso and Y. N. Patt. Decision procedures for surjectivity and injectivity of parallel maps for tesselation structures. *Journal* of Computer and System Sciences, 6:448–464, 1972.
- [2] A. Dennunzio and E. Formenti. Decidable properties of 2d cellular automata. *Lecture Notes in Computer Science*, 5257:264–275, 2008.
- [3] G. Hedlund. Endomorphisms and automorphisms of shift dynamical systems. *Mathematical Systems Theory*, 3:320–375, 1969.
- [4] J. Kari. Reversibility and surjectivity problems of cellular automata. Journal of Computer System Science, 48:149–182, 1994.
- [5] A. Maruoka and M. Kimura. Condition for injectivity of global maps for tessellation automata. *Information and Control*, 32:158– 162, 1976.
- [6] D. Richardson. Tessellation with local transformation. *Journal of Computer System Science*, 6:373–388, 1972.