Exact Capacity of Wireless Multihop Networks

Maher Heal
Abu Dhabi University
Email: maher.hashem@adu.ac.ae

***Abstract*—Finding the maximum flow, or capacity, of wireless multihop networks received a considerable attention by the research community due to its importance from theoretical and practical standpoints. However, since it is np-hard, only bounds can be found using different heuristics. In this poster we find an upper bound on the maximum flow supported by static wireless multihop networks for any load matrix and arbitrary topology in a polynomial time, via a Linear Program, by exploiting only a local interference conflict graph. By noting that interference is local, we replace the optimization problem condition of listing maximal independent sets by listing maximal cliques which improves computational complexity of the solution. Doing this, we had a quadratic programming problem to calculate the exact maximum flow, and not bounds, for any network which is polynomial for some interference models such as the Unit Disk Graph. We applied our model to an example network in [1] and obtained the exact maximum network flow, while [1] suggests only a solution to calculate bounds for the network. The model was also applied to other networks such that the conflict-graph is not perfect, where other models fail to calculate the capacity, and we were able of obtaining the exact capacity.**

Keywords - Capacity, maximum flow, conflict graph, linear programming, quadratic programming.

I. Introduction

The reader is referred to Jain's paper [1] for details. Here, we just write the optimization problem and our modification by taking conflicts (interference) into consideration locally.

We start with a wireless network of N nodes that are connected by unidirectional links L. The vertices in the network graph correspond to the wireless nodes N and the edges correspond to the wireless links L between nodes. The optimization problem has two parts: Part I which is same as that in wired networks and Part II is a restriction due to conflict because of interference and media contention.

Part I:
max $\sum_{l_{si} \in L} f_{si}$
Subject to:

$$\sum_{l_{ij} \in L} f_{ij} = \sum_{l_{ji} \in L} f_{ji} \quad n_i \in N \backslash \{n_s, n_d\}$$

$$\sum_{l_{is} \in L} f_{is} = 0$$

$$\sum_{l_{di} \in L} f_{di} = 0$$

$$f_{ij} \le C_{ij} \quad \forall i,j | l_{ij} \in L, f_{ij} \ge 0 \quad \forall i,j | l_{ij} \in L \quad (1)$$

The optimization problem above is for maximizing the flow from source node s to destination node d. Due to interference and media contention, we cannot route data on every single path in wireless multi-hop networks. We can route only on paths of conflict-free links. Using the conflict graph of the network we have Part II below.

Part II:

$$\sum_{i=1}^{K'} \lambda_i \le 1 \quad (2)$$

$$f_{ij} \le \sum_{l_{ij} \in I_i} \lambda_i . C_{ij} \quad (3)$$

Here λ_i is the fraction of time allocated to maximal independent set I_i and K' is the total number of maximal independent sets in the conflict graph. The number of maximal independent sets may be exponentially many as per [1].

A. A local conflict graph formulation

Focusing on each link conflicts locally, let the fraction of time link l_{ij} is active is γ_{ij} and let l_1, l_2, ..., l_k be the conflicting links with link l_{ij} as per the conflict graph. Assume γ_1, γ_2, ...,γ_k be the fraction of active times for links l_1, l_2, ..., l_k respectively. Then we have the following restriction on the flow. Note that the flow may constitute successful and conflicting flows.

$$0 \le \gamma_{ij} \le 1 \quad \forall i,j | l_{ij} \in L \quad (4)$$

$$(1 - \gamma_1 - \gamma_2 - ... - \gamma_k).C_{ij} \le f_{ij} \le (1 - max(\gamma_1, \gamma_2, ..., \gamma_k)).C_{ij} \quad \forall i,j | l_{ij} \in L \quad (5)$$

Eq. 4 is clear. Eq. 5 states that maximum flow f_{ij} can be more than left side when all the conflicting links send data and are active at different times but no more than the right side when all the conflicting links are sending data at the same time. The non-linear constraint 5 can be converted to linear by substituting $max(\gamma_1, \gamma_2, ..., \gamma_k)$ by say z_{ij}. Then Eq. 5 can be written as:

$$(1 - \gamma_1 - \gamma_2 - ... - \gamma_k).C_{ij} \le f_{ij} \le (1 - z_{ij}).C_{ij} \quad \forall i,j | l_{ij} \in L \quad (6)$$

$$\gamma_i \le z_{ij} \quad \forall i = 1, 2, ..., k \quad (7)$$

We seek to maximize the flow from the source node s. Let us say there are m links emanating from the source node labeled $l_1, l_2, ..., l_m$ and the flow of these links and their active times are $f_1, f_2, ..., f_m$ and $\gamma_1, \gamma_2, ..., \gamma_m$ respectively. Then if we assume the network is up for a period of time T, it is clear

978-1-4673-9795-7/15 $31.00 © 2015 IEEE
DOI 10.1109/CSCI.2015.108

total successful flow from source node s is less than or equal what is given by:

$$[f_1.\gamma_1 T + f_2.\gamma_2 T + ... + f_m.\gamma_m T]/T = f_1.\gamma_1 + f_2.\gamma_2 + ... + f_m.\gamma_m \quad (8)$$

We have equality when all the flows of all m emanating links are not conflicting during their entire activity times (transmission times). Our optimization problem is now maximizing the upper bound in non-linear Eq. 8 given the linear constraints in Eqs. 1, 4, 6, and 7. However, by noting that $f_i = \gamma_i.C_i, i = 1, 2, ..., m$ we have, Eq. 8 equals to:

$$f_1^2/C_1 + f_2^2/C_2 + ... + f_m^2/C_m \quad (9)$$

so our problem is to maximize Eq. 9 over the polyhedron determined by the set of linear constraints 1, 4, 6, and 7.

It can be shown the values of $f_1, f_2, ..., f_m$ of the solution are the same if we replace our objective function in Eq. 9 by the following linear function:

$$f_1 + f_2 + f_2 + ... + f_m \quad (10)$$

Applying our formulation to the example network in [1] of 3×3 lattice grid, and by using the sufficient Row constraints in [2] to show the upper bound is a scheduleable, we obtained exactly the same optimum flow of 0.5 and the same exact schedule of the links in a polynomial time. Only an upper bound of 0.667 was found in [1]. To the Peterson graph [3] in Fig. 1, we obtained an upper bound of 12 while the independence number is 9.

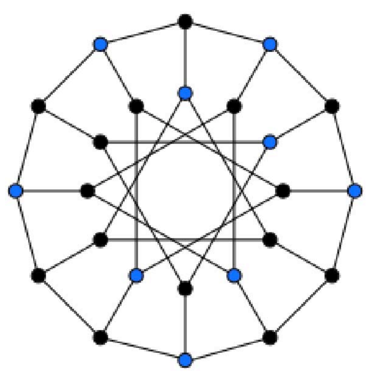

Fig. 1. A Peterson Graph has independence number = 9.

B. Exact Maximum Flow - Quadratic Programming Model

For each link l_j, let ζ_j, $0 \leq \zeta_j \leq 1$, be the fraction of the transmission time γ_j that the transmission was successful on link l_j, $1 \leq j \leq |L|$. Hence, assuming there are m links emanating from the source, the throughput (successful flow) is given by:

$$throughput = f_1.\gamma_1.\zeta_1 + f_2.\gamma_2.\zeta_2 + ... + f_m.\gamma_m.\zeta_m \quad (11)$$

Moreover, we have the following additional constraint:
For each node in the conflict graph (each link) we have:

$$\sum \gamma_{p_i}\zeta_{p_i} + ... + \gamma_{r_i}\zeta_{r_i} \leq 1 \quad \forall i = 1 \ to \ q \quad (12)$$

where q is the number of maximal cliques containing the node and $p_i, ..., r_i$ are the nodes in the maximal clique i that the node is a member of , $1 \leq i \leq q$. This is true because the total successful time in each in each maximal clique is less than or equal 1. Now, replacing each $\gamma_i\zeta_i$, $i = 1$ to $|L|$ by θ_i, $0 \leq \theta_i \leq 1$, we have the following complete nonlinear optimization problem for flow maximization:

Max :$f_1.\theta_1 + f_2.\theta_2 + ... + f_m.\theta_m$, m links emanating from source node s
subject to in addition to Eq. 1 :

$$\sum \theta_{p_i} + ... + \theta_{r_i} \leq 1 \quad (13)$$

for each node where q is the number of maximal cliques containing the node and $p_i, ..., r_i$ are the nodes in clique i, $1 \leq i \leq q$

$$\theta_i \leq \gamma_i \quad \forall i | i \in L \quad (14)$$

$$(1-\gamma_1-\gamma_2-...-\gamma_k).C_{ij} \leq f_{ij} \leq (1-z_{ij}).C_{ij} \quad \forall i,j | l_{ij} \in L \quad (15)$$

$$\gamma_i \leq z_{ij} \quad \forall i = 1, 2, ..., k \quad (16)$$

$$0 \leq \gamma_i \leq 1, 0 \leq \theta_i \leq 1 \quad \forall i | i \in L \quad (17)$$

C. Illustrative Examples and Results

The quadratic programming model was applied to Jain's network example discussed in previous section and we were able to get a capacity of 0.5 (a capacity of 0.5625 is sometimes was obtained depending on the initial starting point of the solver). Two more examples were used to verify the model. First one, is Peterson Graph, [3], see Fig. 1. As it is known finding the independence number of a graph is equivalent to calculating the capacity of a two nodes network that has a conflict matrix equivalent to the graph. For the Peterson graph shown in Fig. 1, the independence number is 9 which is exactly what we obtained by using our model. The last example is the largest graph with a known independence number in the database of House of Graphs [4], a graph of 200 vertices and 2200 edges and independence number 100. We obtained the exact independence number in almost 5 minutes; 1 minute or less for listing the maximal cliques in the graph and almost 4 minutes or less for the matlab SQP solver to converge. It is clear the Peterson Graph is non-perfect due to odd holes. Finally, through it is clear using maximal cliques has a better computation time due to local connectivity of the conflict graph, polynomial time is possible for certain graph such as the Unit Disk Graph.

REFERENCES

[1] K. Jain, J. Padhye, V. N. Padmanabhan, and L. Qiu, "Impact of interference on multi-hop wireless network performance," in *Proceedings of the 9th Annual International Conference on Mobile Computing and Networking*, ser. MobiCom '03. New York, NY, USA: ACM, 2003, pp. 66–80. [Online]. Available: http://doi.acm.org/10.1145/938985.938993

[2] R. Gupta, J. Musacchio, and J. Walrand, "Sufficient rate constraints for qos flows in ad-hoc networks," *Ad Hoc Networks*, vol. 5, no. 4, pp. 429–443, 2007.

[3] Wikipedia. Independent set (graph theory). Visited on 4-11-2015. [Online]. Available: https://en.wikipedia.org/wiki/Independent_set_(graph_theory)

[4] House of Graphs. Visited on 4-11-2015. [Online]. Available: http://hog.grinvin.org/