Reconstruction of Uniform Sampling from Nonuniform Sampling Using Discrete Cosine Transform

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Abstract — Reconstruction of uniform sampling from nonuniform sampling using the DCT is described. It was shown by experiment that the reconstruction using the DFT of the symmetrically extended sequence improved the performance greatly as the symmetric extension avoids discontinuity that adds high-frequency content [1]. The DCT of a sequence is equivalent to the DFT of the symmetrically extended sequence. In this paper, the relationship between the DCT of a uniformly sampled sequence and the DCT of a nonuniformly sampled sequence is derived. Using the relationship an algorithm to reconstruct uniform sampling from nonuniform sampling has been developed.

Keywords — uniform sampling, nonuniform sampling, DFT, DCT

I. INTRODUCTION

Reconstruction of a uniformly sampled sequence from a nonuniformly sampled sequence using the DCT is described in this paper. Computing the DFT of a signal is the same as computing the Fourier series coefficients of the periodically extended signal. If the extended signal has discontinuity at the junction of extension, it will unduly add high frequency content and increase the bandwidth of the extended signal especially when the signal is very short.

To prevent such discontinuity, the symmetric extension has been considered [1]. It has been shown by experiment that the reconstruction using the half-sample symmetric extension performed the best. Instead of using the DFT of the symmetrically extended signal we propose to use the DCT for the reconstruction. Because DCT does not need doubling of the sequence length and complex operations, the new method requires less computational complexity and would be more desirable. Out of four possible DCT techniques we used the DCT-2 [2]. The DCT-2 of a sequence is identical to the DFT of the half-sample symmetrically extended sequence.

The paper is organized as follows. In section II, the relationship between the DFT of a uniformly sampled sequence and the DFT of a nonuniformly sampled sequence is obtained and the perfect reconstruction condition is explained. In section III, the relationship between the DCT of a uniformly sampled sequence and the DCT of a nonuniformly sampled sequence is obtained. From the relationship, the formula to reconstruct the DCT of the uniformly sampled sequence from the DCT of the nonuniformly sampled sequence is derived when the nonuniform sampling ratios are known. In section IV, reconstruction experiments are presented. Finally, a conclusion is made in section V.

II. RELATIONSHIP BETWEEN UNIFORM SAMPLING AND NONUNIFORM SAMPLING

Uniform sampling means that a continuous-time signal, \( x(t) \), is sampled uniformly at \( t = 0, T, 2T, \ldots, (N-1)T \) where the sampling interval or period, \( T \), is constant. Nonuniform sampling means that the sampling interval is not constant as shown in Fig. 1. Throughout this paper we assume that the number of samples taken between 0 and \( NT \) [s] is \( N \) for nonuniform sampling so that the average sampling interval is \( T \). The nonuniform sampling ratios, \( a_n \), are assumed to be known parameters.

Let \( x_p(t) \) be the periodic signal that is obtained by periodically extending \( x(t) \) with the period of \( NT \) [s] where \( N \) is an odd integer. When \( N \) is even, a similar procedure can be followed. The periodic signal, \( x_p(t) \), will have its Fourier series with the fundamental radian frequency of \( 2\pi/NT \) [rad/s] as shown in equation (1) if the periodically extended signal has no harmonics greater than \((N−1)/2\):

\[
 x_p(t) = \sum_{k=-N/2}^{N/2-1} F_k e^{j2\pi nk/T} \tag{1}
\]

where \( F_k \) are the Fourier series coefficients and \( j = \sqrt{-1} \).
If \( N \) samples of \( x_p(t) \) are taken uniformly between 0 and \( NT \) with the sampling interval, \( T \), then equation (1) becomes

\[
x_p(nT) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} F_k e^{j \frac{2\pi k n}{N}} \text{ for } n = 0, 1, 2, \ldots, N-1.
\] (2)

Because \( x_p(t) \) is identical to \( x(t) \) for \( 0 \leq t < NT \), equation (2) can be rewritten as a sequence

\[
x(n) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} F_k e^{j \frac{2\pi k n}{N}} \text{ for } n = 0, 1, 2, \ldots, N-1.
\] (3)

Now equation (3) can be rewritten as

\[
x(n) = \sum_{k=0}^{N-1} \frac{X(k)}{N} e^{j \frac{2\pi k n}{N}} \text{ for } n = 0, 1, 2, \ldots, N-1.
\] (4)

where

\[
X(k) = \begin{cases} 
F_k & \text{for } 0 \leq k \leq \frac{N-1}{2} \\
F_{k-N} & \text{for } \frac{N-1}{2}+1 \leq k \leq N-1
\end{cases}
\] (5)

Let us define the vectors as follows.

\[
x = [x(0), x(1), x(2), \ldots, x(N-1)]^T
\] (6)

\[
w_k = [1, e^{j \frac{2\pi k}{N}}, e^{j \frac{2\pi (k+1)}{N}}, \ldots, e^{j \frac{2\pi (N-1)k}{N}}]^T
\] (7)

From equation (4), one can show that \( x \) can be expressed in terms of a linear combination of \( w_k \)'s so that

\[
x = \sum_{k=0}^{N-1} X(k) w_k
\] (8)

Now \( X(k)/N \) in equation (8) is the \( w_k \) component of \( x \). The \( X(k)/N \) can be computed by the projection of \( x \) onto \( w_k \) so that

\[
\frac{X(k)}{N} = \frac{w_k^* x}{w_k^* w_k} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi k n}{N}}
\] (9)

where the superscript * denotes the complex conjugation transpose. Equation (9) is known as the DFT formula:

\[
X(k) = \sum_{n=0}^{\frac{N-1}{2}} x(n) e^{-j \frac{2\pi kn}{N}} \text{ for } k = 0, 1, 2, \ldots, N-1.
\] (10)

By plugging \( x(n) \) of equation (4) into equation (10), one obtains

\[
X(k) = \sum_{m=0}^{\frac{N-1}{2}} \left[ \frac{1}{N} \sum_{n=0}^{N-1} x(m) e^{j \frac{2\pi m n}{N}} \right] e^{-j \frac{2\pi kn}{N}}
\] (11)

If \( N \) samples are taken nonuniformly between 0 and \( NT \), the expression of the nonuniformly sampled sequence becomes

\[
\tilde{x}(n) = x_p \left( (n + \alpha_n)T \right) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} F_k e^{j \frac{2\pi n (n+\alpha_n)k}{N}}
\] (12)

where \( \alpha_n \) are termed the nonuniform sampling ratios. Equation (12) can be rewritten as

\[
\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j \frac{2\pi n (n+\alpha_n)}{N}}
\] (13)

By replacing \( n \) in the exponent inside the brackets in equation (11) with \( n+\alpha_n \) or by plugging equation (13) into equation (10), the DFT of the nonuniformly sampled sequence is expressed as follows.

\[
\tilde{X}(k) = \sum_{m=0}^{\frac{N-1}{2}} \left[ \frac{1}{N} \sum_{n=0}^{N-1} X(m) e^{j \frac{2\pi m (n+\alpha_n)k}{N}} \right] e^{-j \frac{2\pi kn}{N}}
\] (14)

The order of the summations in equation (14) can be reversed so that

\[
\tilde{X}(k) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{j \frac{2\pi m (n+\alpha_n)k}{N}} e^{-j \frac{2\pi kn}{N}} X(m)
\] (15)

Reconstruction of uniform sampling from nonuniform sampling using the DFT is as follows. First, find the DFT, \( \tilde{X}(k) \), of the nonuniformly sampled signal, \( \tilde{x}(n) \). Second, estimate the DFT, \( X(k) \), of the uniformly sampled signal using the relationship in equation (15). Finally, reconstruct \( x(n) \) by taking the IDFT of the estimation of \( X(k) \) [1].

When \( N \) is odd, for perfect reconstruction the highest harmonic of \( x_p(t) \) should not be greater than \( (N-1)/2 \). When \( N \) is even, for perfect reconstruction the highest harmonic of \( x_p(t) \) should not be greater than \( N/2 \). In other words, for perfect reconstruction using the DFT the bandwidth of \( x_p(t) \) should be less than \( \pi/T \) [rad/s] (which is \( N/2 \) multiplied by \( 2\pi/NT \)).
III. RECONSTRUCTION USING DCT

The following continuous-time signal is considered in this and the next sections.

\[ x(t) = e^{-0.1t} \cos(0.2\pi t)u(t) \]  

where \( u(t) \) is the unit step function. The sampling interval \( T = 1 \) [sec] and the number of samples, \( N = 8 \).

As shown in the previous section, computation of the DFT of a sequence is in fact the same as computing the Fourier series coefficients of the periodically extended signal. The periodic extension is shown in Fig. 2 (a). Note that there is discontinuity at the junction of extension. This discontinuity or sudden jump unduly adds substantial high-frequency contents and hence increases the bandwidth of the extended signal.

Fig. 2. Extension of \( x(n) = e^{-0.1n} \cos(0.2\pi n) \) for \( 0 \leq n \leq 7 \) using (a) periodic extension (b) half-sample symmetric extension followed by periodic extension.

To prevent such discontinuity, symmetric extension is used. It was shown that the half-sample symmetric extension performed the best for the reconstruction using the DFT [1]. The half-sample symmetric extension is shown in Fig. 2 (b). Instead of using the DFT of the symmetrically extended signal we propose to use the DCT for the reconstruction. The half-sample symmetric extension corresponds to the DCT-2 [2]. Suppose a continuous-time signal, \( x(t) \), is sampled at \( t = 0, T, 2T, \ldots, (N-1)T \) where \( T \) is the sampling interval. The DCT of the uniformly sampled sequence, \( x(n) \), for \( n = 0, 1, 2, \ldots, N-1 \), is given by

\[ X_{dct}(k) = \beta(k) \sum_{n=0}^{N-1} x(n) \cos \left( \frac{\pi(2n+1)k}{2N} \right) \]  

(17)

where \( \beta(0) = \frac{1}{\sqrt{N}} \) and \( \beta(k) = \sqrt{\frac{2}{N}} \) for \( k = 1, 2, \ldots, N-1 \). The IDCT is given by

\[ x(n) = \sum_{k=0}^{N-1} \alpha(k)X_{dct}(k) \cos \left( \frac{\pi(2n+1)k}{2N} \right) \]  

(18)

By plugging \( x(n) \) of (18) into (17), one obtains

\[ X_{dct}(k) = \beta(k) \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \beta(m)X_{dct}(m) \cos \left( \frac{\pi(2n+1)m}{2N} \right) \cos \left( \frac{\pi(2n+1)k}{2N} \right) \]  

(19)

Suppose the signal is nonuniformly sampled so that the nonuniformly sampled sequence is given by (as in (12))

\[ x(n) = x(n + \alpha_n)T \]  

(20)

By replacing \( n \) inside the brackets in (19) with \( n + \alpha_n \), the DCT of the nonuniformly sampled sequence is expressed as follows:

\[ X_{dct}(k) = \beta(k) \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \beta(m)X_{dct}(m) \cos \left( \frac{\pi(2n+2\alpha_n+1)m}{2N} \right) \cos \left( \frac{\pi(2n+1)k}{2N} \right) \]  

(21)

The order of the summations in (21) can be reversed so that

\[ X_{dct}(k) = \sum_{m=0}^{N-1} \beta(m) \left[ \beta(k) \sum_{n=0}^{N-1} \cos \left( \frac{\pi(2n+2\alpha_n+1)m}{2N} \right) \cos \left( \frac{\pi(2n+1)k}{2N} \right) \right] X_{dct}(m) \]  

(22)

Let

\[ r(n, m) = \cos \left( \frac{\pi(2n+2\alpha_n+1)m}{2N} \right) \]  

(23)

then equation (22) becomes

\[ \tilde{X}_{dct}(k) = \sum_{m=0}^{N-1} \beta(m)R(k, m)X_{dct}(m) \]  

(24)

where the sequence \( \{ R(0, m), R(1, m), \ldots, R(N-1, m) \} \) is the DCT of the sequence \( \{ r(0, m), r(1, m), \ldots, r(N-1, m) \} \) for \( m = 0, 1, \ldots, N-1 \).

In matrix form, equation (24) becomes

\[ \tilde{X}_{dct} = RDX_{dct} \]  

(25)
where

\[
X_{dtr} = \begin{bmatrix}
X_{dtr}(0) \\
X_{dtr}(1) \\
\vdots \\
X_{dtr}(N-1)
\end{bmatrix}, \quad \tilde{X}_{dtr} = \begin{bmatrix}
\tilde{X}_{dtr}(0) \\
\tilde{X}_{dtr}(1) \\
\vdots \\
\tilde{X}_{dtr}(N-1)
\end{bmatrix},
\]

\[
R = \begin{bmatrix}
R(0,0) & R(0,1) & \cdots & R(0,N-1) \\
R(1,0) & R(1,1) & \cdots & R(1,N-1) \\
\vdots & \vdots & \ddots & \vdots \\
R(N-1,0) & R(N-1,1) & \cdots & R(N-1,N-1)
\end{bmatrix}, \quad \text{and}
\]

\[
D = \text{diag}(\beta(0), \beta(1), \cdots, \beta(N-1)).
\]

By performing the following matrix computation, the DCT of the uniformly sampled sequence can be reconstructed from the DCT of the non-uniformly sampled sequence.

\[
X_{dtr} = (RD)^{-1} \tilde{X}_{dtr}
\] (26)

IV. EXPERIMENTAL RESULTS

For our experiment, the signal of equation (16) is sampled nonuniformly. Statistically independent zero-mean Gaussian random numbers were used for nonuniform sampling ratios, \(a_n\). The standard deviations were chosen as 0.01, 0.02, 0.04, 0.08, 0.16 and 0.32.

(i) Reconstruction using DFT without symmetric extension

The DFT of the nonuniformly sampled sequence is computed and the DFT of the uniformly sampled sequence is estimated using equation (15) [1]. The IDFT of the estimated DFT is computed for reconstruction of the uniformly sampled sequence. 5,000 trials were performed at each standard deviation. The average signal to noise ratio is computed using the following method.

\[
\text{average SNR (in dB)} = 10 \log_{10} \frac{1}{5000} \sum_{n=1}^{5000} \frac{\text{signal power}}{\text{noise power in each trial}}
\] (27)

where signal power = \(\frac{1}{N} \sum_{n=0}^{N-1} x^2(n)\) and

noise power in each trial = \(\frac{1}{N} \sum_{n=0}^{N-1} (x(n) - \hat{x}(n))^2\),

where \(\hat{x}(n)\) is the reconstructed uniformly sampled sequence by taking the IDFT of the estimated DFT, \(X(k)\), obtained according to the reconstruction algorithm [1] in each trial.

(ii) Reconstruction using DCT

The DCT of the nonuniformly sampled sequence is computed and the DCT of the uniformly sampled sequence is estimated using equation (26).

<table>
<thead>
<tr>
<th>Table I</th>
<th>Comparison of performance between the DFT method without symmetric extension and the DCT method. 5000 trials were performed at each standard deviation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation of nonuniform sampling ratios</td>
<td>Average SNR [dB]</td>
</tr>
<tr>
<td></td>
<td>DFT method (without extension)</td>
</tr>
<tr>
<td>0.01</td>
<td>44.0675 dB</td>
</tr>
<tr>
<td>0.02</td>
<td>37.9453 dB</td>
</tr>
<tr>
<td>0.04</td>
<td>31.8171 dB</td>
</tr>
<tr>
<td>0.08</td>
<td>25.4370 dB</td>
</tr>
<tr>
<td>0.16</td>
<td>17.7753 dB</td>
</tr>
<tr>
<td>0.32</td>
<td>too much error</td>
</tr>
</tbody>
</table>

The IDCT of the estimated DCT is computed for the reconstruction of the uniformly sampled sequence. The DCT method performed much better as shown in Table I. There is at least 20-dB advantage in SNR with the DCT method over the DFT method without symmetric extension.

V. CONCLUSION

In this paper reconstruction of a uniformly sampled sequence from a nonuniformly sampled transient sequence using the DCT is described. The DCT method that does not need doubling of the sequence length and complex operations requires less computational complexity and is more desirable than the method based on DFT with symmetric extension.

VI. REFERENCES
