Abstract—Since recent development and dissemination of ICTs activate information exchange on social networks, the dynamics for describing propagation of activities on the networks has become an interesting research object. This paper proposes an oscillation model describing the propagation of activities on social and information networks. In order to analyze such dynamics, we generally need to model asymmetric interaction between nodes. This paper discusses a symmetric matrix-based model that can describe some types of link asymmetry. Although the proposed model is simple, it can reproduce well-known indices of node centrality and can be considered as the underlying mechanism of network dynamics. As an application of the proposed model, we show a framework to estimate natural frequency of networks by utilizing resonance.

Keywords—Laplacian matrix, coupled oscillators, node centrality, resonance

I. INTRODUCTION

Information exchange on social networks is being activated by the popularity of information networks. So, complex dynamics for describing propagation of activities on the social and information networks is a rich source of research topics. In complex network analysis, there are a lot of indices that can describe the characteristics of networks, including degree distribution, clustering coefficient, and many kinds of node centralities [1], [2], [3].

Spectral graph theory is a key approach for investigating the structure of networks [4], and the eigenvalues and the eigenvectors of the Laplacian matrix are important when investigating network structure. Spectral graph theory is applicable to many problems including clustering of networks, graph drawing, graph cut, node coloring, and image segmentation [4]. One of the most significant properties of spectral graph theory is the fact that we can introduce graph Fourier transformation [6], [7], which is the diagonalization of the Laplacian matrix. The advantage of graph Fourier transformation can be found in its ability to decompose network dynamics into scales appropriate for the network’s structure. As a result, complex network dynamics can be understood as the superposition of simple dynamics for each Fourier mode, and network dynamics can be completely understood algebraically.

However, the decomposition of dynamics into Fourier modes is effective only if the Laplacian matrix is symmetric. This is because symmetric matrices always can be diagonalized. User dynamics on a social or information networks is generated by the interaction of nodes on the networks. This interaction is generally asymmetric. In other words, the actions between nodes depend on the direction of links. To represent asymmetric actions on links, directed graphs are frequently used. Since the structure of a directed graph is normally expressed by an asymmetric matrix, graph Fourier transformation cannot be applied.

One proposal on spectral graph theory for directed graphs transforms asymmetric Laplacian matrices in Jordan canonical form via elementary transformation [8], [9]. However, since asymmetric Laplacian matrices cannot always be diagonalized, decomposition of the dynamics into simple Fourier modes remains unavailable.

This paper focuses on some types of link asymmetry that can be represented as node characteristics, and represents the structure of a directed graph by a symmetric scaled Laplacian matrix. In addition, we analyze oscillation dynamics on networks to describe the propagation of activities on directed networks by using symmetric scaled Laplacian matrices.

Typical examples of the asymmetric interaction of links include the relationship between a popular blogger and the followers. The strength of the interaction between them depends on the direction of links, and the strength of activity propagation on links is asymmetric. However, link directionality in this case can be reduced to node characteristics. Furthermore, since similar relations frequently appear in human relations, we expect that various asymmetric links on networks can be analyzed in terms of node characteristics. By using a symmetric matrix to model asymmetric links, we can apply graph Fourier transformation based on the symmetric scaled Laplacian matrix and thus analyze oscillation dynamics on asymmetric networks. Our framework adopts the mass of the node as the node characteristic.

In our model, oscillation dynamics on directed networks can be expressed by the equation of motion of the harmonic oscillator for each Fourier mode. Since the phase of the oscillation cannot be determined by the equation of motion, oscillation dynamics may exhibit complicated behavior that inhibits any intuitive understanding. Our solution is to use the oscillation energy of each node, a phase-free index, to

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Oscillation Model for Network Dynamics Caused by Asymmetric Node Interaction Based on the Symmetric Scaled Laplacian Matrix

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represent the strength of node activity. In simple cases, the oscillation energy can reproduce well-known node centralities of the degree centrality and the betweenness centrality. In general, the oscillation energy depends on the propagation attributes of network activity. Therefore, the oscillation energy is an extended notion of the well-known node centrality. So, we can expect that the proposed oscillation model is the underlying mechanism of activity propagation on networks.

Since the oscillation energy can be measured as the strength of node activity, the way of the usage of the measured value of energy is important for applications. We introduce models that describe the damped oscillation and the forced oscillation on networks. As an application, we propose a method for estimating the eigenvalues of the scaled Laplacian matrix; called the network resonance method. The network resonance method can estimate the eigenvalues by applying resonance of the forced oscillation on networks even if components of the scaled Laplacian matrix is not known.

This paper is organized as follows. In Section II, after defining the Laplacian matrix for directed networks, we introduce a scaled Laplacian matrix that allows asymmetric node interactions to be described by a symmetric matrix. In Sec. III, we analyze oscillation models to describe the propagation of node activity on networks by using the scaled Laplacian matrix. In Sec. IV, we propose the oscillation energy of each node as an extended index of node centrality and discuss the relationship to the well-known node centralities. In Sec. V, we propose the network resonance method to estimate eigenvalues of the scaled Laplacian matrix. Finally, we conclude this paper in Sec. VI.

II. SCALED LAPLACIAN MATRIX FOR DESCRIBING ASYMMETRIC LINK DIRECTION

A. Definition of the Laplacian Matrix

Network structure is frequently expressed as a matrix. Let us consider loop-free directed graph \( G \) with \( n \) nodes. Let the set of nodes be \( V = \{1, 2, \ldots, n\} \) and the set of directed links be \( E \). In addition, let the link weight for link \( (i \rightarrow j) \in E \) be \( w_{ij} > 0 \). We define the following \( n \times n \) square matrix \( A = [A_{ij}] \) as:

\[
A_{ij} := \begin{cases} w_{ij} & ((i \rightarrow j) \in E), \\ 0 & ((i \rightarrow j) \notin E). \end{cases}
\]  

This matrix represents link presence and weights, and is called the (weighted) adjacency matrix.

Next, we define the weighted out-degree, \( d_i \), of node \( i \) \( (i = 1, 2, \ldots, n) \) as:

\[
d_i := \sum_{j \in \partial i} w_{ij},
\]  

where \( \partial i \) denotes the set of nodes adjacent to node \( i \). Also, weighted out-degree matrix \( D \) is defined as:

\[
D := \text{diag}(d_1, d_2, \ldots, d_n).
\]

If all link weights are \( w_{ij} = 1 \) for \( (i \rightarrow j) \in E \), \( d_i \) is reduced to out-degree, i.e. the number of outgoing links from node \( i \).

Based on the above preparation, we define the Laplacian matrix \( \mathcal{L} \) of directed graph \( G \) as follows [4], [5].

\[
\mathcal{L} := D - A.
\]  

The Laplacian matrix is also called the graph Laplacian.

B. Symmetrization of Laplacian Matrix and the Scaled Laplacian Matrix

Let us consider left eigenvectors \( \lambda \) and their eigenvalues \( \lambda \) as:

\[
\lambda \mathcal{M} = \lambda \mathcal{M}.
\]  

If there is a (left) eigenvector \( \lambda \mathcal{M} = \lambda \mathcal{M} \) associated with eigenvalue \( \lambda = 0 \), and \( m_i > 0 \) satisfies:

\[
m_i w_{ij} = m_j w_{ji} \quad (\equiv k_{ij}),
\]  

then the link asymmetry of \( \mathcal{L} \) can be expressed by using a symmetric Laplacian matrix. Note that the oscillation dynamics discussed in the following sections satisfies these conditions. The procedure to represent \( \mathcal{L} \) by a symmetric matrix is shown as follows. First, we consider a directed graph and introduce its Laplacian matrix \( L = \mathcal{D} - A \), where \( A = [A_{ij}] \) is defined as:

\[
A_{ij} := \begin{cases} k_{ij} & ((i, j) \in E), \\ 0 & ((i, j) \notin E), \end{cases}
\]  

and \( D = \text{diag}(\sum_k A_{ik}, \sum_j A_{kj}, \ldots, \sum_j A_{nj}) \). Since \( k_{ij} = k_{ji} \), \( L \) is a symmetric Laplacian matrix for a certain undirected graph. By using \( L \), the asymmetric Laplacian matrix \( \mathcal{L} \) is expressed as:

\[
\mathcal{L} = M^{-1} L,
\]  

where \( M := \text{diag}(m_1, m_2, \ldots, m_n) \) means the scaling factors of nodes. Figure 1 shows a simple example of the above procedure: where \( w_{ij} = k_{ij} / m_i \) is decomposed into \( 1 / m_i \) and \( k_{ij} \).

Here, we introduce the scaled Laplacian matrix that is defined as:

\[
S := M^{-1/2} L M^{-1/2}.
\]  

Note that \( S \) is a symmetric matrix. Let \( x = (x_1, x_2, \ldots, x_n) \) be a (right) eigenvector associated with an eigenvalue \( \lambda \), that is, \( \mathcal{L} x = \lambda x \). By multiplying \( M^{1/2} \) to the eigenvalue equation from the right, we obtain:

\[
M^{1/2} \mathcal{L} x = S (M^{1/2} x) = \lambda (M^{1/2} x).
\]  

![Fig. 1](image.png)

An example of the Laplacian matrix for a directed graph and its symmetrization.
This means the scaled Laplacian matrix $S$ has the same eigenvalues of $L$, and its eigenvector is $y := M^{1/2} x$. Since the quadratic form of $S$ is

$$b y S y = \sum_{(i,j) \in E} k_{ij} \left( \frac{y_i}{m_i} - \frac{y_j}{m_j} \right)^2 \geq 0,$$

the eigenvalues of $S$ (also $L$) are nonnegative. Let us sort the eigenvalues in ascending order,

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}.$$

We can choose eigenvector $v_{\mu}(\mu = 0, 1, \ldots, n-1)$ as the orthonormal eigenvector associated with $\lambda_{\mu}$. That is,

$$S v_{\mu} = \lambda_{\mu} v_{\mu}, \quad v_{\mu} \cdot v_{\nu} = \delta_{\mu \nu}, \quad (10)$$

where $\delta_{\mu \nu}$ denotes the Kronecker delta.

III. OSCILLATION MODELS ON NETWORKS

A. Oscillation Model Based on Asymmetric Interactions

To describe the propagation of activity of a node through networks, let us consider oscillation dynamics on networks. The relationship between oscillating phenomena and well-known indices for network dynamics is be discussed in Sec. IV.

Let weight $x_i$ of node $i$ be displacement from the equilibrium, and let its restoring force be proportional to the difference in the displacements of adjacent nodes. Figure 2 is a representative image of our oscillation model. Although the figure shows a 1-dimensional network, it is easily extended to general networks. To represent diverse oscillating behavior, we allow the spring constant of each link to be different and the mass of each node to also be different.

Here, it is worthy to note about the validity of oscillation model whose restoring force is proportional to the difference of displacements. Let the restoring force of node $i$ be a function $f(\Delta x)$ of the difference $\Delta x := x_i - x_j$ of the displacements of adjacent nodes $i$ and $j$. It is natural to assume $f(\Delta x) = 0$ if $\Delta x = 0$. For small $\Delta x$, we can expand $f(\Delta x)$ as

$$f(\Delta x) = -k_{ij} \Delta x + O(\Delta x^2),$$

where $k_{ij}$ is a positive constant corresponding to the spring constant. So, our oscillation model can be considered as the basic and universal model if nonlinear effects in $O(\Delta x^2)$ are relatively small.

Incidentally, there is a well-known oscillation model, called the Kuramoto model (Fig. 3)[10]. This model consists of the same (or similar) oscillators coupled by weak interaction, and mainly describes the synchronization of these oscillators. Thus our oscillation model differs from Kuramoto model.

We assign a spring constant to each link and express it as link weight $k_{ij} > 0$. In addition, we assign mass $m_i > 0$ to each node $i$. Let $x_i$ be the displacement of node $i$ and $p_i$ be its conjugate momentum. Then, Hamiltonian $H$ of our coupled oscillator system is expressed as

$$H := \sum_{i \in V} \frac{(p_i)^2}{2m_i} + \sum_{(i,j) \in E} \frac{k_{ij}}{2} (x_i - x_j)^2 = \sum_{i \in V} \frac{(p_i)^2}{2m_i} + \frac{1}{2} (x L x).$$

Fig. 2. Oscillation model on networks.

Fig. 3. Kuramoto model.

By applying canonical formalism, the equations of motion are derived as follows.

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{n} L_{ij} x_j, \quad \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m_i}.$$

By eliminating $p_i$ from these equations, we have the following wave equation as the equation of motion,

$$m_i \frac{d^2 x_i}{dt^2} = -\sum_{j=1}^{n} L_{ij} x_j,$$

or written in vector form as

$$M \frac{d^2 x}{dt^2} = -L x, \quad (11)$$

where $M := \text{diag}(m_1, \ldots, m_n)$ and $x := (x_1, \ldots, x_n)$. By multiplying $M^{-1}$ from the left, we have the equation of motion as

$$\frac{d^2 x}{dt^2} = -M^{-1} L x = -L x; \quad (12)$$

note that it is based on asymmetric interactions. To diagonalize the equation of motion, we introduce vector $y$ which is defined by

$$y = M^{1/2} x,$$

and the equation of motion can be written as

$$\frac{d^2 y}{dt^2} = -S y. \quad (13)$$

It follows that the equation of motion will yield the eigenvalue problem of the symmetric scaled Laplacian matrix, and node mass can be understood as the node scaling factor. Node mass can represent the strength of inertia, and is related to the strength of the asymmetric influence to adjacent nodes. In addition, the spring constant of links can represent the strength of influence between each pair of adjacent nodes. Furthermore, the condition (5) corresponds to Newton’s 3rd
law (about equivalency of the strength between an action and its reaction).

Let \( y = y(t) \) be expanded by the eigenbasis of \( S \), \( \mathbf{v}_\mu \), as
where \( y(t) = \sum_{\mu=1}^{n-1} a_\mu(t) \mathbf{v}_\mu \) and solve the equation of motion for the Fourier mode \( a_\mu(t) \) \((\mu = 0, 1, \ldots, n-1)\). The procedure of expansion by eigenbasis is known as graph Fourier transformation [6], [7]. The solution is given by
\[
\alpha_\mu(t) = c_\mu e^{i(\omega_\mu t + \theta_\mu)},
\]
where \( \omega_\mu^2 = \lambda_\mu \), \( i = \sqrt{-1} \), \( \theta_\mu \) denotes phase, and \( c_\mu \) is a constant. The solution of oscillation on networks (the solution of (12)) is expressed as
\[
x(t) = M^{-1/2} \left( \sum_{\mu=0}^{n-1} c_\mu e^{i(\omega_\mu t + \theta_\mu)} \mathbf{v}_\mu \right).
\]
Note that the phase cannot be determined by the equation of motion, but the oscillation behavior varies widely with the phase. Consequently, to understand the universal aspect of oscillation dynamics, a kind of phase-free index is required. This issue is discussed in Sec. IV.

B. Damped Oscillation Model

In actual situations, any oscillation is damped with time. This subsection shows a model for the damped oscillation on networks.

Let us consider the equation of motion for the damped oscillation
\[
M \frac{d^2 x(t)}{dt^2} + \gamma M \frac{dx(t)}{dt} = -L x(t),
\]
where \( \gamma \) is a constant. Here \( \gamma M \) means the viscous damping coefficient, where it is important to note that the viscous damping coefficient is assumed to be proportional to node mass. By using vector \( y = M^{1/2} x \), we can diagonalize the equation of motion as
\[
\frac{d^2 y(t)}{dt^2} + \gamma \frac{dy(t)}{dt} = -S y(t).
\]
The equation of motion for Fourier mode \( a_\mu(t) \) is expressed as
\[
\frac{d^2 a_\mu(t)}{dt^2} + \gamma \frac{da_\mu(t)}{dt} + \omega_\mu^2 a_\mu(t) = 0,
\]
where \( \omega_\mu^2 = \lambda_\mu \). To analyze the solution of this equation, we assume the solution takes the form of \( a_\mu(t) \propto e^{\alpha t} \). By substituting this into the equation of motion, we obtain the characteristic equation
\[
\alpha^2 + \gamma \alpha + \omega_\mu^2 = 0.
\]
There are three different solutions to the equation of motion according to the solution of the characteristic equation, \( \alpha = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \omega_\mu^2} \). In the case of \( (\gamma/2)^2 < \omega_\mu^2 \), the solution describes damped oscillations,
\[
a_\mu(t) = c_\mu e^{-(\gamma/2)t} \cos \left[ \sqrt{\omega_\mu^2 - (\gamma/2)^2} t + \theta_\mu \right],
\]
where \( c_\mu \) and \( \theta_\mu \) are constants. In the case of \( (\gamma/2)^2 = \omega_\mu^2 \), the solution describes critical damping,
\[
a_\mu(t) = (a_\mu(0) + c_\mu t) e^{-(\gamma/2)t},
\]
where \( c_\mu \) is a constant. Finally, in the case of \( (\gamma/2)^2 > \omega_\mu^2 \), the solution describes overdamping. Let \( \alpha_+ \) and \( \alpha_- \) (both values are negative) denote the solutions of the characteristic equation, the solution of the equation of motion is
\[
a_\mu(t) = c_\mu^+ e^{\alpha_+ t} + c_\mu^- e^{\alpha_- t},
\]
where \( c_\mu^+ \) and \( c_\mu^- \) are constants.

C. Forced Oscillation Model

This subsection introduces a forced oscillation model on networks. Let us consider the situation that we impose forced oscillation with angular frequency \( \omega \) on a certain node, \( j \), as an external force. The equation of motion of the forced oscillation is
\[
M \frac{d^2 x(t)}{dt^2} + \gamma M \frac{dx(t)}{dt} + L x(t) = (F \cos \omega t) \mathbf{1}_j,
\]
where \( F \) is a constant and \( \mathbf{1}_j \) is only the \( j \)-th component that is 1, all other components are 0, that is,
\[
\mathbf{1}_j = (0, \ldots, 0, 1, 0, \ldots, 0).
\]
By using vector \( y = M^{1/2} x \), the equation of motion can be diagonalized as
\[
\frac{d^2 y(t)}{dt^2} + \gamma \frac{dy(t)}{dt} + S y(t) = F \cos \omega t \sqrt{m_j} \mathbf{1}_j.
\]
Since \( y(t) \) depends on \( \omega \), we redefine \( y(\omega, t) := y(t) \). By expanding \( y(\omega, t) \) and \( \mathbf{1}_j \) using the eigenbasis of the scaled Laplacian matrix \( S \), we introduce the Fourier modes \( a_\mu(\omega, t) \) and \( b_\mu \) as
\[
y(\omega, t) = \sum_{\mu=0}^{n-1} a_\mu(\omega, t) \mathbf{v}_\mu, \quad \mathbf{1}_j = \sum_{\mu=0}^{n-1} b_\mu \mathbf{v}_\mu.
\]
The equation of motion of Fourier mode \( a_\mu(\omega, t) \) is expressed as
\[
\frac{\partial^2 a_\mu(\omega, t)}{\partial t^2} + \gamma \frac{\partial a_\mu(\omega, t)}{\partial t} + \omega_\mu^2 a_\mu(\omega, t) = \frac{F \cos \omega t}{\sqrt{m_j}} b_\mu.
\]
The solution of the inhomogeneous equation (25) is the sum of the solutions of the corresponding homogeneous equation (17) and the particular solution of (25). Since the solution of homogeneous equation (17) is damped with time, only the oscillation of the particular solution of (25) remains after some long time. Since the angular frequency of the particular solution should be \( \omega \), the particular solution can be expressed as
\[
a_\mu(\omega, t) = A_\mu(\omega) \cos(\omega t + \theta_\mu)
\]
where
\[
A_\mu(\omega) = A_\mu(\omega) \cos \theta_\mu - \sin \omega t \sin \theta_\mu.
\]
By substituting it into the equation of motion (25), the amplitude $A_m(\omega)$ and phase $\theta_m$ of the particular solution are obtained as

$$A_m(\omega) = \frac{F b_m}{\sqrt{m_j}} \frac{1}{\sqrt{(\omega_m^2 - \omega^2)^2 + (\gamma \omega)^2}}, \quad \tan \theta_m = \frac{-\gamma \omega}{\omega_m^2 - \omega^2}.$$  \hfill (27)

IV. NODE CENTRALITY

As shown in Sec. III-A, the wave equation (12) cannot describe the phase of oscillations. Since the behavior of oscillating phenomenon has extremely different appearance if the phase changes, it is hard to extract useful information from direct observation of oscillating aspects. Of course, since $a_m(t)$ of (14) is a complex-valued function, the value of $a_m(t)$ cannot be observed in actual networks. This section introduces the oscillation energy of each node as a non-negative-valued phase-free index, and shows that it can reproduce the well-known indices of node centrality. This means that our oscillation model can be considered as an underlying mechanism of the propagation of activities on networks.

For the oscillation model described in Sec. III-A, we define node activity as the oscillation energy of the node. From (14), the amplitude of the Fourier mode $a_m(t)$ is $c_m = |d_m(t)|$. In addition, let $v_m$ be the eigenbasis associated with the eigenvalue $\lambda_m$ of scaled Laplacian matrix $S$, and let its components be expressed as

$$v_m = (v_m(1), v_m(2), \ldots, v_m(n)).$$

Since the oscillation of node $i$ is the superposed oscillations for Fourier modes of the node, the oscillation energy $E_i$ of node $i$ is obtained by summing the oscillation energy for each Fourier mode, as

$$E_i = \frac{1}{2} \sum_{\mu=0}^{n-1} \omega_m^2 (c_m v_m(i))^2.$$  \hfill (28)

To demonstrate the calculation of the oscillation energy of each node, we use the network model shown in Fig. 4, where all the link weights are set at 1. As the initial condition of the wave equation (12), we can give the displacement only at a certain node. We call the node as a source node of activity. First of all, let us consider the situation that the source node of activity is chosen at random. In this case, all the Fourier modes contribute at the same strength. Figures 5 (a) and (b) show the oscillation energy of each node for different source nodes, 1 and 12, respectively, where the scaling factors is chosen as $M = I$. The results show that the oscillation energy strongly depends on the source node of activity. Therefore, the oscillation energy is changed not only by network topology, but also node mass (Fig. 5) and the propagation scenario of activity on networks (Fig. 6). In other words, the oscillation energy also depends on link asymmetry, and strength and location distributions of source nodes. Since the oscillation energy is reduced to the well-known degree node centrality in the simplest case, the oscillation energy for each node can be understood as an extended notion of the degree centrality.

The betweenness centrality is another well-known node centrality. Let the number of shortest paths between node $j$ and node $k$ be $\sigma_{jk}$, and the number of those paths passing through the node $i$ be $\sigma_{jk}(i)$. The betweenness centrality $g(i)$ for node $i$ is defined as

$$g(i) := \sum_{j, k \in V} \frac{\sigma_{jk}(i)}{\sigma_{jk}}.$$  \hfill (29)

The normalized betweenness centrality $\bar{g}(i)$ is defined as

$$\bar{g}(i) := \frac{2 g(i)}{n(n-1)(n-2)}.$$  \hfill (30)

Asymmetric node interaction. From Fig. 5 (a), we can recognize that the oscillation energy is proportional to the node degree centrality (the oscillation energy for each node is proportional to its node degree). So, Fig. 5 (b) can be regarded as an extension of node degree centrality considering asymmetry of node interaction. If a certain specific node is the source of activity, node oscillation energy would be quite different. Figures 6 (a) and (b) show the oscillation energy of each node for different source nodes, 1 and 12, respectively, where the scaling factors is chosen as $M = \hat{D}^2$. The results show that the oscillation energy strongly depends on the source node of activity. Therefore, the oscillation energy is changed not only by network topology, but also node mass (Fig. 5) and the propagation scenario of activity on networks (Fig. 6). In other words, the oscillation energy also depends on link asymmetry, and strength and location distributions of source nodes. Since the oscillation energy is reduced to the well-known degree node centrality in the simplest case, the oscillation energy for each node can be understood as an extended notion of the degree centrality.

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$$\bar{g}(i) := \frac{2 g(i)}{n(n-1)(n-2)}.$$  \hfill (30)
The physical meaning of $g(i)$ is the ratio of the number of shortest paths including node $i$ to the number of combination of node pairs in $V \setminus \{i\}$, that is $(n - 1)(n - 2)/2$.

Next, we set the link weight $k_{ij}$ of the network model shown in Fig. 4 as the number of the shortest paths passing through the link $(i,j)$. Figure 7 (a) shows the difference between the oscillation energy $E_i$ for each node and the minimum energy $E_{\text{min}}$ defined as

$$E_{\text{min}} := \min_{i \in V} E_i.$$  

Figure 7 (b) shows the normalized betweenness centrality $\bar{g}(i)$ for each node. We can recognized that the difference between the oscillation energy is proportional to the betweenness centrality. From the same reason for degree centrality, the oscillation energy $E_i$ gives an extension of the well-known betweenness centrality.

The oscillation energy gives extensions of node centralities even if we consider the damped oscillation on networks. Detailed discussion is presented in [11].

V. NETWORK RESONANCE METHOD FOR INVESTIGATING THE EIGENVALUES OF NETWORK DYNAMICS

Since the actual structure of a network is difficult to know, it is almost impossible to measure components of the scaled Laplacian matrix $S$, directly. For example, in social networks, the strength and significance of friendships (links) are hard to observe. Thus the eigenvalues of $S$, the key to describing the oscillation dynamics on networks, cannot be calculated from $S$. However, since the oscillation energy is related to the node centrality that is the strength of activity of node on networks, we probably be able to measure the oscillation energy as a real solid object. The oscillation energy is related to the natural frequency and the amplitude. In this section, we discuss a way to estimate natural frequency (square root of eigenvalue) of $S$ from observation of the amplitude that is obtained from observation of the oscillation energy.

As recognized from discussion in Sec. III-C, amplitude $A_{\mu}(\omega)$ of (27) takes maximal value at

$$\omega = \sqrt{\omega_\mu^2 - \gamma^2/2}.$$  

This phenomenon is called the resonance. When we observe the oscillation of a node caused by forced oscillation, the mixture of oscillation (26) for each $\mu$, that is, $y(\omega; t)$ of the first equation of (24) is observed. We propose a method to estimate eigenvalue $\lambda_\mu$ (or $\omega_\mu = \sqrt{\lambda_\mu}$) and damping factor $\gamma$ from observations of the amplitude $A(\omega) := |y(\omega; t)|$ of the response oscillation (Fig. 8). In actual, the amplitude $A(\omega)$ is indirectly obtained from observations of oscillation energy.

The Q-factor represents the sharpness of amplitude $A_{\mu}(\omega)$ with respect to $\omega$. On both sides of the peak of amplitude $A_{\mu}(\omega)$, we define frequencies $\omega_\mu^+$ and $\omega_\mu^-$ that give the amplitudes $A_{\mu}(\omega_\mu^+)$ and $A_{\mu}(\omega_\mu^-)$ that are $1/\sqrt{2}$ times the peak value of $A_{\mu}(\omega)$ ($\omega_\mu^- < \omega_\mu^+$). Since oscillation energy is proportional to the square of the amplitude, $\omega_\mu^+ - \omega_\mu^-$ means the half width for energy. The Q-factor is defined as

$$Q_\mu := \sqrt{\frac{\omega_\mu^2 - \gamma^2/2}{\omega_\mu^+ - \omega_\mu^-}}.$$  

We assume $\gamma \ll \omega_\mu$ and approximate $A_{\mu}(\omega)$ around $\omega = \omega_\mu$. By using $\omega^2 - \omega_\mu^2 \simeq 2\omega_\mu(\omega - \omega_\mu)$,

$$A_{\mu}(\omega) \simeq \frac{F b_{\mu}}{\sqrt{m_j}} \frac{1}{\sqrt{(2\omega_\mu(\omega - \omega_\mu))^2 + (\gamma \omega_\mu)^2}} = \frac{F b_{\mu}}{\sqrt{m_j} \omega_\mu} \frac{1}{\sqrt{4(\omega - \omega_\mu)^2 + \gamma^2}},$$  

(28)

Therefore,

$$A_{\mu}(\omega_\mu) \simeq \frac{F b_{\mu}}{\sqrt{m_j} \omega_\mu} \gamma, \quad A_{\mu}(\omega_\mu \pm \gamma/2) \simeq \frac{1}{\sqrt{2}} A_{\mu}(\omega_\mu),$$  

and we have $\omega_{\mu}^+ = \omega_\mu \pm \gamma/2$ (double-sign indicates correspondence). Consequently, we have

$$Q_\mu \simeq \frac{\omega_\mu}{\gamma}.$$  

(29)

These relations enable us to estimate natural frequency $\omega_\mu$ (or the eigenvalue $\lambda_\mu = \omega_\mu^2$) and damping factor $\gamma$.

We use the network model shown in Fig. 4, where all link weights are 1 and node mass is also set to $M = 1$. Figures 9 (a) and (b) show examples of network resonance for external force input by node 1 and 12, respectively: the amplitude $A(\omega) = |y(\omega; t)|$ is observed at node 1 (red line) and node 10 (blue line) as the response of the external force with angular frequency $\omega$. Depending on the pair of input node and observed node selected, the amplitude $A(\omega)$ exhibits a different aspect. Therefore, we expect that eigenvalues of the scaled Laplacian matrix can be estimated from appropriate pairs of input and observed nodes.
of usage of the measured value of energy is important. We proposed a network resonance method that can estimate the eigenvalues of the scaled Laplacian matrix and the damping factor, from measurements.

We also expect that this method can estimate the absolute value of the component of eigenvectors. If the sign of the components are determined by orthogonal condition of eigenvectors, we obtain pairs of eigenvalues and eigenvectors of the scaled Laplacian matrix, from measurements. This means the original Laplacian matrix (including the weight of directed link and network topology) can be reproduced. So, our framework is applicable to investigate network structure that is not observed directly; for example, social networks of users, networks of malicious hosts generating cyber attacks, etc.

In security application, we probably can use the framework of [12], for example. First, we access malicious web site with the frequency of $\omega$. These accesses induce that malicious users attack to a honeypot, and we observe their response. The framework corresponds to the network resonance method based on forced oscillation.

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