Exponential Synchronization of Lur’e Networks with Stochastic Disturbances via Impulsive Control

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Abstract—In this paper, the exponential synchronization of the stochastic Lur’e networks with nonlinear and directed couplings is investigated. Through the impulsive control strategy, a criterion is derived to guarantee the realization of the exponential synchronization by means of the Lyapunov stability theorem and the comparison principle.

Keywords: Lur’e network, impulsive controller, synchronization.

1. Introduction

Among the collective behaviors of complex networks, synchronization is one of the most important behaviors which plays an important role in practical applications \cite{1-2}. The common and effective control strategies are pinning control, impulsive control, intermittent control and so on. For example, \cite{3} analyzed a kind of complex dynamical network with time delayed coupling and non-delayed coupling. It is well known that impulse is a common phenomenon in many evolving networks. In the impulsive communication framework, the dynamics of each node is only affected by its neighbors at the impulsive instants and there exist impulsive effects in the dynamical behavior of nodes. Good prior works had been made on the synchronization of complex works with impulses controllers, see \cite{4-6} and the references therein. For instance, \cite{6} investigated the global exponential synchronization of delayed complex dynamical networks with nonidentical nodes and stochastic perturbations through adaptive control and impulsive control schemes. In this paper, the outer exponential synchronization of stochastic networks will be discussed. The dynamics of every node in the network studied in this paper follows the Lur’e system which has nonlinear local dynamical behaviors. In order to simulate the real world, we take the stochastic phenomenon into consideration since the instantaneous perturbations and abrupt changes exist in many realistic networks. The information interchange between nodes is in a nonlinear and asymmetrical way. By using impulsive control strategy, a sufficient condition is obtained to guarantee the realization of the exponential synchronization for all initial values by means of the Lyapunov stability theorem and the comparison principle.

2. Problem description

In this section, some preliminaries will be firstly given. Then, we give the complex dynamical Lur’e network model with stochastic perturbations with nonlinear coupled functions and an asymmetrical coupled matrix.

**Definition 1** (\cite{7}) The average impulsive interval of the impulsive sequence $\chi = \{t_1, t_2, \cdots \}$ is less than $T_a$, if there exist a positive integer $N_0$ and a positive number $T_a$ such that $N_0(T, t) \geq \frac{T-a}{T-a} - N_0, \forall T \geq t \geq 0$ where $N_0(T, t)$ denotes the number of impulsive times of the impulsive sequence $\chi$ in the time interval $(t, T)$.

Consider the following network model with nonlinear coupling, and the nodes’ dynamics follows Lur’e system

$$dx_i(t) = [Ax_i(t) + B\tilde{f}(Cx_i(t))]dt + \tilde{p}(x_i(t), t)dw(t) + e\sum_{j=1}^{N}g_{ij}\Gamma H(x_j(t))dt$$

(1)

where $x_i(t) = [x_i^1(t), x_i^2(t), \cdots, x_i^n(t)]^T \in \mathcal{R}^n$, for $i = 1, 2, \cdots, N$. $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{n \times n}$ are constant matrices. The constant $c > 0$ denote the coupling strength and $\Gamma = \text{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\} \in \mathcal{R}^{n \times n}$ is the inner-linking matrix with $\gamma_i \geq 0$. Function $\tilde{f} : \mathcal{R}^m \rightarrow \mathcal{R}^m$ is memoryless nonlinear vector valued function which is continuously differentiable on $\mathcal{R}$, $\tilde{f}(0) = 0$. $G = (g_{ij}) \in \mathcal{R}^{N \times N}$ is the the coupling matrix and $g_{ij} > 0$ if there is a connection from node $i$ to node $j(i \neq j)$, otherwise $g_{ij} = 0$ and it satisfies zero-sum-row condition and $g_{ij} \neq g_{ji}$, which means the network we study is a directed information transmission network. The nonlinear coupling functions $\Gamma H(x)(t) = (\tilde{h}(x_i^1(t)), \tilde{h}(x_i^2(t)), \cdots, \tilde{h}(x_i^n(t)))^T$, satisfies the following conditions:

$$\frac{\tilde{h}(u)-\tilde{h}(v)}{u-v} \geq \theta > 0,$$

for any $u, v \in \mathcal{R}$, $w(t) \in \mathcal{R}^m$ is an $m$-dimensional Brownian motion, $\tilde{p} : \mathcal{R}^n \times \mathcal{R}^+ \rightarrow \mathcal{R}^{n \times m}$ is the noise intensity function matrix satisfying $\tilde{p}(t, 0^+) = 0^{n \times m}$.

The salve system with impulsive control scheme can be described as

$$dy_i(t) = [Ay_i(t) + B\tilde{f}(Cy_i(t))]dt + \tilde{p}(y_i(t), t)dw(t) + e\sum_{j=1}^{N}g_{ij}\Gamma H(y_j(t))dt,$$

(2)
We control the first $l$ nodes in the Lur’e network with the following impulsive control strategy

$$u_i(t) = \sum_{k=1}^{\infty} \mu[y_i(t) - x_i(t)]\delta(t - t_k),$$

(3)

where the constant $\mu \in (-2, 0)$, which means the corresponding impulsive effects are synchronizing, the time series $\chi = \{t_1, t_2, \cdots, t_n, \cdots\}$ is a sequence of strictly increasing impulsive instants satisfying $t_{k-1} < t_k$ and $\lim_{k \to \infty}(t_k) = +\infty$. And $\delta(\cdot)$ is the Dirac impulsive function, which satisfies $\int_{-\infty}^{+\infty} \delta(t) dt = 1$. Define $e(t) = y(t) - x(t)$, the error system can be written as

$$de_i(t) = \left[ Ae_i(t) + B f(Ce_i(t)) \right]dt + p(e_i(t), t)dw(t) + \sum_{j=1}^{N} g_{ij} H(e_j(t))dt \quad t \neq t_k,$n

$$e_i(t_k^+) = e_i(t_k^-) + \mu e_i(t_k^-) \quad t = t_k,$$

and denote $f(Ce_i(t)) = f(Cy_i(t)) - f(Cx_i(t))$, $p(e_i(t), t) = \tilde{p}(y_i(t), t) - \tilde{p}(x_i(t), t)$, $h(e_i(t), t) = \tilde{h}(y_i(t), t) - \tilde{h}(x_i(t), t)$, $H(e_i(t)) = [h(e_1^T(t)), h(e_2^T(t)), \cdots, h(e_N^T(t))]^T$, for $i = 1, 2, \cdots, N$; $j = 1, 2, \cdots, n$.

In order to derive the main results of the stochastic impulsive dynamical Lur’e network, we make the following hypothesis.

**Assumption 1:** There exists a constant $\zeta > 0$ such that $\|f(u) - f(v)\| \leq \zeta \|u - v\|$ holds for any $u, v \in \mathbb{R}^n$.

**Assumption 2:** Assume the function matrix $\tilde{p} : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n \times M}$ satisfy trace$[(\tilde{p}(u) - \tilde{p}(v))^T(\tilde{p}(u) - \tilde{p}(v))] \leq M(\|u - v\|^2$ for any $u, v \in \mathbb{R}^n$, where $M$ is a known constant matrix with compatible dimensions.

3. Main results

In the following part, we investigate the global and exponential synchronization of the synchronization. Throughout the paper, we always assume that $\xi_i(t_k) = \xi_i(t_k^-)$. Let $\xi = (\xi_1, \xi_2, \cdots, \xi_N)^T$ be the left unitization eigenvector of $G$ corresponding to the eigenvalue 0.

**Theorem 1:** Suppose that Assumption 1 and 2 hold and the average impulsive interval of the impulsive sequence $\chi = \{t_1, t_2, \cdots\}$ is less than $T_a$. Let $\rho = (1 + \mu)^2, \mu \in (-2, 0)$, and consider the impulsive stochastic dynamical master Lur’e network (1) and the slave Lur’e network (2). Then the controlled impulsive stochastic dynamical network (4) is globally and exponentially stable if the following inequality is satisfied

$$\ln \frac{\rho}{T_a} + \lambda_{\max}(A + AT + M^TM) + 2\zeta \sqrt{\lambda_{\max}(B^TB)\lambda_{\max}(C^TC)} \leq 0,$$

(5)

**Proof:** From a Lyapunov function $V(t) = \sum_{i=1}^{N} \xi_i^T(t)e_i(t)$, and its $L\mathcal{V}(t)$([5]), it can be verified the theorem. The detail is omitted.

**Example:** The network (1) and (2) have the parameters as

$$A = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 0 \\ 0 & -15 & -0.0385 \end{bmatrix}, B = \begin{bmatrix} 5.9 \\ 0 \\ 0 \end{bmatrix},$$

and $C = [1, 1, 1]$, $f(Cx_i(t)) = \frac{1}{2}(|x_i(t)| + 1 - |x_i(t) - 1| \cdot dw(t)$ is an 3-D Brownian motion, $\tilde{p}(x_i(t), t) = 0.5 \cdot |x_i(t)| \cdot I_3$, $h(x(t)) = 8x(t) + 0.2\sin(x(t))$. We consider a network consists of 4 nodes. For given parameters, we derive $\rho = 0.040, \delta = \lambda_{\max}(A + AT + M^TM) + 2\zeta \sqrt{\lambda_{\max}(B^TB)\lambda_{\max}(C^TC)} = -321.8876$, and $\ln \frac{\rho}{T_a} + \delta = -293.1626 < 0$. From the above analysis, the above parameters could let the networks achieve synchronization.

In Fig. 1, we plot the estimation of boundaries of synchronization region and the synchronization error curve.

[Fig. 1: (a) Synchronization region with respect to $\mu$, $T_a$. (b) Error evolution curve $E(t)$.]

**Acknowledgement:** The work was supported by Basic Science Research Program through the NRF funded by the Ministry of Education (2013R1A1A2A0005201).

**References**


