Time Adapivity Stable Finite Difference for Acoustic Wave Equation

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Abstract—Acoustic wave modeling is widely used to synthesize seismograms theoretically, being the basis of the reverse time migration strategy. Explicit Finite Difference Method (FDM) is often employed to find numerical solution of this problem and in this case, spatial discretization is related to the shortest wavelength to be captured and temporal discretization is determined by stability condition. In this case small grid size has to be used to assure a stable and accurate solution and algorithms with locally adjustable time steps can be of advantageous use when treating heterogeneous domains. In this paper, we are concerned with a temporal adaptivity algorithm: Region Triangular Transition algorithm (RTT), discussing its accuracy and its computational efficiency when applied to complex heterogeneous domains. To evaluate computational efficiency of this algorithm we present, in this work, how computational cost varies with the subregions size ratio of the heterogeneous medium when compared with computational cost of the conventional algorithm using only one time step, showing how this adaptivity algorithm can be used in complex domains to reduce the amount of values to be calculated. Some discussion are made concerning how dispersion error can be reduced when adaptive schemes are used.

Keywords: time adaptivity, finite difference, seismic analysis

1. Introduction

Acoustic wave modeling is widely used to theoretically synthesize seismograms and is also the basis of the reverse time migration. Explicit Finite Difference Method (FDM) is frequently used in the numerical solution of this problem and spatial discretization is related to the shortest wavelength to be captured and temporal discretization is restricted by some stability condition. A small grid size helps to increase the accuracy but the substantial growth in memory and computational cost may become these methods prohibitive for realistic modeling.

Several proposals have been presented to improve the accuracy and stability of the FDM, including staggered-grid finite-difference method \cite{1, 2}, variable grid difference schemes \cite{3}, higher order operators to approximate the derivatives \cite{4, 5}, and variable time step.

In the application of FDM in heterogeneous media in its conventional form, to ensure numerical stability of the solution, temporal discretization is given by the smallest time step defined by the subregion of the heterogeneous domain with the highest wave propagation velocity. In this context algorithms using locally adjustable time steps have been proposed to optimize computational cost as it occurs in numerical seismic modeling, adjusting the time increment for each subregion with different physical characteristics of the considered domain.

Falk et al. \cite{6} proposed, in 1998, an algorithm for local time step adjustment for the solution of elastic wave equation in a domain composed of two subregions, obtaining a reduction of 77\% in process time if compared to conventional algorithm. Meanwhile in this algorithm the time steps needed to be proportional to $2^n$, being $n$ an integer value, a new algorithm was suggested by Tessmer \cite{7}, extending this proportional limiting relationship to any integer value $n$. Although these formulations work well in heterogeneous domains, reducing the computing effort, accuracy may be affected with the appearance of a noise in the signal wave which was identified for time steps close to the stability threshold of the used method as shown in \cite{8} and in \cite{9}. In these references we presented a new adjustable time step algorithm, named RTT, that remains stable even when using time steps close to the stability condition for each sub-region of the heterogeneous domain. The amount of reduction in computational cost is directly related to the domain to be analyzed, either through the relationship between the values of different existing propagation velocities as well as with the size of each subdomain associated with these propagation velocities.

To evaluate computational efficiency of this algorithm we present in this work how computational cost varies with the heterogeneous medium subregions sizes ratio when compared to the computational cost of the conventional algorithm using only one time step. We show also, with a complex domain example how this adaptivity algorithm can be used in such cases to reduce the amount of values to be calculated. Some discussion is made concerning how dispersion error can be reduced when adaptive schemes are used.
2. Acoustic Wave Equation

The acoustic wave equation for a two-dimensional (2D) problem can be described as:

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f(x, y, t) \tag{1}
\]

where \( u = u(x, y, t) \) is a function of space and time and \( c \) is the medium wave velocity propagation, \( f(x, y, t) \) the time variation of the source distribution. Initial conditions are \( u(x, y, 0) = g(x, y) \) and \( \frac{\partial u}{\partial t}(x, y, 0) = h(x, y), \forall x, y \in \Omega \) and suitable boundary conditions are adopted.

For the two-dimensional acoustic wave, using Finite Difference Method (FDM) with temporal and spatial second order approximations \((2-2)\) one obtain an explicit method with the following expression:

\[
\begin{align*}

u_{i,j}^{t+\Delta t} &= 2u_{i,j}^t + C^2 (u_{i+1,j}^t - 2u_{i,j}^t + u_{i-1,j}^t) \\
&+ u_{i-1,j}^t + u_{i,j+1}^t - 2u_{i,j}^t + u_{i,j-1}^t) - \\
& - u_{i,j}^{t-\Delta t} + \Delta t^2 f(x, y, t) + O(\Delta t^2, h^2)
\end{align*}
\]

with Courant number \( C = \frac{c\Delta t}{\Delta x} \), where \( t \) designates the actual time, \( i \) and \( j \) indexes designate spatial mesh points in directions \( x \) and \( y \) while \( \Delta t \) and \( h \) define time and spatial discretizations respectively, with \( \Delta x = \Delta y = h \). Next we can see in Eq. 3 the expression of the explicit Finite Difference Method (FDM) when using time second order and spatial fourth order approximations \((2-4)\).

\[
\begin{align*}

u_{i,j}^{t+\Delta t} &= 2u_{i,j}^t + \frac{C^2}{12} \left[ -u_{i+1,j}^{t+2} + u_{i,j}^t - \\
&- u_{i-1,j}^t - u_{i,j+2}^t - u_{i,j-2}^t + \\
&+ 16(u_{i+1,j}^t + u_{i-1,j}^t + u_{i,j+1}^t + u_{i,j-1}^t - 6u_{i,j}^t) - u_{i,j}^{t-\Delta t} + \\
&+ \Delta t^2 f(x, y, t) + O(\Delta t^2, h^4) \right].
\end{align*}
\]

The finite difference approximations previously described are conditionally stable and their stability limits for homogeneous domains given in terms of Courant number \( C \), can be found in [10].

Conventional explicit finite-difference algorithms in heterogeneous media are based on a global and constant time step to ensure the stability of the method. The use of locally adjustable time steps can help to save a great deal of computing time, using intermediate increments in time that are adjusted depending on the local wave velocity propagation. In this case, the choice of these intermediate time intervals, must satisfy the stability limits for each subregion of the heterogeneous domain.

Close to these subregions separating boundaries, spatial derivatives approximations have to be calculated by accessing mesh points of finite differences that could not have their values defined in the same instant of time due to the use of different temporal discretizations. These points work like ghostpoints for subregions using these intermediate time steps. The way proposed to calculate these values in this transition region, is that differentiate these adaptive algorithms. Thus, the transition region depends on the algorithm used and of finite difference order to approximate the spatial derivatives. To preserve accuracy, temporal adaptivity algorithms use finite difference approximation of the same order to calculate values for the points in the transition region but differ in how to calculate these values.

As mentioned in Section 1, Falk et al. [6] proposed an algorithm where time increments needed to be multiples of \( 2^n \), where \( n \) is a positive integer while Tessmer [7] suggested a modification to this algorithm which lets you use any previous time discretization satisfying \( \Delta t_{\text{subreg}} = \Delta t/(n+1) \). This last algorithm has a constant transition zone width, which varies only with the order of the finite difference approximation used to approximate the spatial derivatives. Both these algorithms utilize different time steps \((\Delta t)\) in the definition of the transition region, ranging between some maximum \( \Delta t_{\text{max}} \) and the minimum \( \Delta t_{\text{min}} \) and produce good results when discretizations are far below the stability limit of the methods [6], [7], [8]. However, in these schemes a noise appears in the solution close to the border of the physical discontinuities for temporal discretizations near to the stability limit of the method, and this noise spread throughout the domain over time [9]. In [8] we proposed an algorithm named RTT that uses the same time steps in all transition region and its efficiency is analyzed here in the present paper.

2.1 Region Triangular Transition Algorithm (RTT)

The Region Triangular Transition (RTT) temporal adaptivity algorithm we proposed in [9], enables the adoption of the same time steps ratio subdivision presented by Tessmer algorithm in [7] with the advantage of the attenuation of the undesirable effects at the interface where wave speed changes.

The RTT algorithm construction is shown in Fig. 1 to Fig. 3 considering time second-order and space fourth order approximation FDM (named RTT 2-4) applied to a 1D domain composed of two sub-regions: Region1 with propagation velocity \( V_1 \) and locally stable time step \( \Delta t \) and mesh points represented by black triangles and retangles and Region2 with velocity \( V_2 = 4V_1 \) and time step \( \Delta t/4 \), and mesh points indicated by white retangles.

Although with this scheme the transition region is enlarged compared to others adaptive schemes it preserves the same time discretization to calculate all points values that are directly used to approximate values near two subregions separating boundary. This transition zone forms a triangular
region composed by the black triangles in the four intermediate times between \( t \) and \( t + \Delta t \) as shown in Fig. 1, since we set here \( V_2 = 4V_1 \).

Points in Fig. 2 and Fig. 3 show the new values to be calculated in time \( t + \Delta t \). Points of the transition region which are calculated in the first level of time \( t + \Delta t/4 \), require values in two moments of earlier times and that are outside the triangular region. Only those values of the transition region are calculated \( \Delta T = 3 \Delta t/4 \), as shown in Fig. 3.

This scheme can be easily generalized to any integer ratio between different time steps, as \( \Delta t \) and \( \Delta t/2 \) or \( \Delta t \) and \( \Delta t/3 \); for combinations of different temporal discretizations provided time steps have integer ratios, as \( \Delta t, \Delta t/2 \) and \( \Delta t/4 \) or \( \Delta t, \Delta t/3 \) and \( \Delta t/6 \); and also for different spatial dimensions (2D and 3D domains).

### 2.2 Considerations about Dispersion Error

In FDM the analysis of the dispersion error are made considering homogeneous media taking constants spatial and temporal discretizations. In this section we present results to show what influence the use of different time discretizations can cause, in dispersion error. To illustrate this let us take the 1D scalar acoustic wave equation:

\[
\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}
\]

with properly boundary and initial conditions.

For explicit FDM with temporal and spatial second order approximations (2-2) we obtain the following expression:

\[
\begin{align*}
\mathbf{u}_i^{t+\Delta t} & = 2\mathbf{u}_i^t + C^2(\mathbf{u}_{i+1}^t - 2\mathbf{u}_i^t + \mathbf{u}_{i-1}^t) \\
& - \mathbf{u}_i^{t-\Delta t} + O(\Delta t^2, h^2).
\end{align*}
\]

and dispersion error [11] is given by:

\[
\omega = \frac{C}{\Delta t} \sin \left[ \frac{c}{\Delta x} \sin \left( \frac{\Delta x}{2} \right) \right].
\]

One obtains a better view about dispersion error using normalized group velocity defined as \( \nu_g = \frac{1}{c} \frac{d\omega}{dk} \) and \( k = \frac{k\Delta x}{c} \), which applied to Eq. 5 produces:
\[ \frac{v_y}{c} = \cos(K) \left[ 1 - c^2 \frac{\Delta t^2}{\Delta x^2} \sin^2(K) \right]^{-1/2} \tag{6} \]

Alford [12] showed that dispersion on finite differences is smaller for Courant numbers close to stability limit and we explore this result on our analysis. Although this analysis treats only homogeneous domains it was used here to evaluate the dispersion error behaviour in two domains using the same spatial discretization assuming one domain has a wave velocity propagation twice the other \((V_1 = 2V_2)\). As an example Fig. 4 shows dispersion error results when we use the same time step in both regions which means different Courant numbers such as \(C_1 = 0.8\) and \(C_2 = 0.4\). We can see in this figure different dispersion errors and note more dispersion on the region with the lower velocity. One can expect that, in this case, the dispersion error in one region influences the results for the whole domain. This result suggests a smaller dispersion on adaptive algorithms in comparison with classical algorithm when the time step is adjusted to keep the lowest dispersion error which, in this case, is \(\Delta t_2 = 2\Delta t_1\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dispersion.png}
\caption{Dispersion in classic algorithm}
\end{figure}

3. Numerical Results

3.1 Comparing RTT with Conventional Algorithm

To compare results obtained with the RTT algorithm with conventional FDM and reference results [8], Equation (1) was solved in a 2D square \((\Omega)\), consisting of an heterogeneous field, subdivided in two subregions: in \textit{Subregion1} \((\Omega_1)\), we assumed the lower propagation velocity, \(V_1 = 750\) m/s while \textit{Subregion2} \((\Omega_2)\) had the greatest velocity, \(V_2 = 4V_1 = 3000\) m/s. The interface with velocity discontinuity were at \(x = 3250\) m and initial condition was given by:

\[ u(x, 0) = \frac{1}{12800\pi} e^{\left(-\frac{(x^2+y^2)}{12800}\right)} \tag{7} \]

and \(\frac{\partial u}{\partial t}(x, y, 0) = 0, \forall x, y\). As boundary condition we considered homogeneous Dirichlet, ie, \(u(\Gamma, t) = 0, t > 0\), being \(\Gamma\) the boundaries of the considered domain, with \(x \in [0; 6000]\) and \(y \in [0; 6000]\). To stress the effects of RTT algorithm in the numerical result precision we show results obtained with different spatial discretizations, including coarse meshes.

Figure 5, Figs. 6 and 7 show results in \(y = 4500\) m and Fig. 8, Figs. 9 and 10 show results in \(x = 4500\) m both at \(t = 0.90\) s for finite difference method with time second order and spatial fourth order approximations(2-4). We consider \(h = \Delta x = \Delta y = 10\) m; 20 m and 40 m with \(C = 0.61\), (close to the stability limit), using conventional FDM taking this Courant number for the subregion where the propagation velocity is \(V_2 = 4V_1\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{solution_y_4500.png}
\caption{Solution in \(y = 4500\) m at \(t = 0.90\) s and \(h = 10\) m}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{solution_x_4500.png}
\caption{Solution in \(x = 4500\) m at \(t = 0.90\) s and \(h = 20\) m.}
\end{figure}

It can be seen, for all displayed graphics, that RTT algorithm gives a better approximation to the adopted reference result than those obtained with conventional methods.
3.2 Computational Cost Comparison

Temporal adaptivity always reduces the number of floating-point operations and this reduction depends on the properties of each heterogeneous media considered, being directly related to the subregions size ratios and to their wave propagation velocities relation. To evaluate the efficiency of RTT algorithm we show in Tables 1 and 2, results obtained for the same 2D domain described in Section 3.1 varying the interface subregion boundary. The first column shows the position ($y$) of the interface between these two subregions, $\Omega_1$ and $\Omega_2$ while second and third columns present the percentages of the whole domain that each subregion occupies. Processing times comparisons are displayed as the ratio of RTT time processing to conventional algorithm time processing, for each one of these six cases. These results are shown in columns 4 and 5 of these tables: on Table 1 when using the RTT 2-2 and on Table 2 for RTT 2-4 scheme. Both algorithms, conventional and RTT, used mesh sizes with $h = 10$ m and $h = 20$ m and, while RTT adaptive scheme employed two time steps, one for each subregion, conventional algorithm used only the smallest time step of that in RTT scheme.

3.3 Treatment of Complex Domains - Adaptivity Possibilities

Next, to show an even more practical implication of the strategy presented here, we evaluated the use of different time steps combinations possibilities in the adaptivity scheme. To do this we used the well-known 2D synthetic acoustic Marmousi model, shown in Fig. 11 which contains, due to its heterogeneity, a complex 2-D wave velocity distribution field in a square domain with $(4602 \times 1502)$m, where the great wave velocity is 4.3 km/h.

Time adaptivity in this kind of data can be executed by joining distinct subregions, which can have the same step, defining what we call here $\Delta t$'s map as shown in Fig. 12 and Fig. 13. This subdivision depends on the desired adaptivity scheme to be used. The most refined temporal discretization $\Delta t_{min}$, associated to the higher wave velocity propagation, defines all others time increments to be used for others regions defined in this map.

For the comparative analysis, here taken in effect, we used a spatial discretization of $\Delta x = \Delta y = 1m$, totaling 6,912,204 values to be calculated per time step, considering two different temporal discretizations combinations in RTT
any adaptivity one has to adopt more integration steps with conventional algorithms, without considering the number of values needed to be calculated is $4\Delta t/2$ and $2\Delta t/4$. 

For this first configuration one obtains the distribution shown in Fig. 12, $\Delta t_{\text{min}} = \Delta t/2$ and $\Delta t_{\text{max}} = \Delta t = 2\Delta t_{\text{min}}$ while for the second one obtains Fig. 13, $\Delta t_{\text{min}} = \Delta t/4$ and $\Delta t_{\text{max}} = \Delta t = 4\Delta t_{\text{min}}$. In this case we considered $\Delta t_{\text{min}} = 0.000164$ s.

For a simulation time of only 1 s, corresponding to 6098 time integration steps with conventional algorithm, without any adaptivity one has to adopt $\Delta t = \Delta t_{\text{min}}$ and the amount of values needed to be calculated is 42,150,619,992. In Table 3 and Table 4 we present the quantity of values to be evaluated for each time discretization using RTT adaptive scheme for configurations showed in Fig. 12 and Fig. 13 displaying their relations with the number of values to be calculated with conventional FDM.

Table 3: Number of operations - Time-steps used: $\Delta t_{\text{min}}$ and $2\Delta t_{\text{min}}$.

<table>
<thead>
<tr>
<th>Values of $\Delta t$</th>
<th>Number of Intervals</th>
<th>Values number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000164</td>
<td>6098</td>
<td>201,439,626</td>
</tr>
<tr>
<td>0.000328</td>
<td>3049</td>
<td>20,974,580,183</td>
</tr>
<tr>
<td>Total RTT</td>
<td>21,176,039,809</td>
<td></td>
</tr>
<tr>
<td>Perc. values RTT/Conv.</td>
<td>50.24%</td>
<td></td>
</tr>
</tbody>
</table>

4. Conclusion

Acoustic wave equation employed to model problems in seismic modeling applications involving heterogeneous media by conventional processing with explicit finite difference algorithm: (a) $\Delta t$ and $\Delta t/2$; (b) $\Delta t$, $\Delta t/2$ and $\Delta t/4$.

When high contrasts in the values of physical characteristics are present, as it occurs with the variation in wave propagation speed values in the subsurface model, temporal adaptive procedures are a good choice for the reduction of computational effort. Working with big problems these savings with RTT algorithm, can reach considerable values as it was shown here for not so large problem.

We also showed that with RTT we can obtain less dispersion error than using conventional algorithm and exemplify how to apply it in more complex domains with the use of a pre-processing step constructing a $\Delta t$'s map. In this case, pre-processing are required with some computational extra cost, and an efficient algorithm can make this task easily classifying and ranking the different subregions according to their wave speed propagation which, with the spatial mesh characteristics, defines the maximum time step to be
Fig. 13: $\Delta t$ map ($\Delta t_{\text{min}} = \Delta t/4$, $2\Delta t_{\text{min}} = \Delta t/2$ and $4\Delta t_{\text{min}} = \Delta t$)

Table 4: Number of operations - Time-steps used: $\Delta t_{\text{min}}$, $2\Delta t_{\text{min}}$, $4\Delta t_{\text{min}}$

<table>
<thead>
<tr>
<th>Values of $\Delta t$</th>
<th>Number of Intervals</th>
<th>Values number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000164</td>
<td>6698</td>
<td>199,185,072</td>
</tr>
<tr>
<td>0.000328</td>
<td>3049</td>
<td>71,050,849</td>
</tr>
<tr>
<td>0.000656</td>
<td>1524</td>
<td>10,448,905,188</td>
</tr>
<tr>
<td>Total RTT</td>
<td></td>
<td>10,719,141,109</td>
</tr>
<tr>
<td>Perc. values RTT/Convent.</td>
<td></td>
<td>25.43%</td>
</tr>
</tbody>
</table>

used in each one of those. This RTT procedure follows the same methodology for different temporal discretizations possibilities and it can be easily extended to tridimensional domains. Finally as in seismic problems the same model domain is solved several times changing only the seismic source location, this $\Delta t'$ s map can be generate only once for all analysis what makes RTT an attracting algorithm.

References