A fast algorithm for coordinate rotation without using transcendental functions

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Abstract— *Coordinates rotation is widely used in science and engineering. It has applications on astronomy, image processing, robotics, power electronics, etc. This paper presents an efficient algorithm to calculate a rotation transform using digital devices. The method is based on the rational trigonometry. Unlike conventional trigonometry which is based on the concepts of angle and distance, the rational trigonometry is based on the concepts of spread and quadrature. In is also presented an analysis of the number of operation that can be saved using a standard math library in a typical situation.*

Keywords: Coordinate transformation, Rotation matrix, Spread, Rational trigonometry

1. Introduction

Coordinate transform is a key concept in mathematics and a very useful tool in engineering. Particularly, coordinate rotation transform is widely employed in image processing, robotics and power electronics among others industrial and scientific applications [1], [2]. Typical application in these areas could require hundreds of rotation transforms per second. The most standard method to carry out a rotation transform is to multiply the original coordinates times a special matrix called rotation matrix. The elements of such matrix are trigonometric functions of the rotation angle.

Numerical methods are necessary to calculate trigonometric functions using digital devices. In general, first power terms of the power series is employed. For example to calculate $\sin(\alpha)$ the first five terms of its Taylor series can be used if $\alpha \in [-\pi, \pi]$. However, if the angle is not in this interval the error increase rapidly. Hence it is necessary to preprocess the angle before using the Taylor series. Furthermore, calculation of the five terms of Taylor series, depending on the variable type (float, frac, etc.), requires additional calculations to obtain the coefficients for each term.

The fact that evaluation of several trigonometric functions is necessary to calculate a single rotation matrix together with the high number of rotation transforms per second that are necessary to perform in a typical application make the use of high power computing devices mandatory for such applications. In power electronics for example, the need of many rotation transform per second prevent the use of low cost microcontrollers for common industrial applications like inverters, active filters and motor drives. This circumstance could be changed with a more efficient methods to perform a rotation transform. The search for efficient methods to perform coordinates rotation has a long history [3] and still continues. The cordic method for example has attracted many effort in the past decades [3], [4], [5], [6].

Recently, some mathematicians have questioned the need of real numbers in math [7], [8]. They say that all math ideas could be expressed with rationals and some irrational numbers. Seeking to solve trigonometrical problems without using real numbers the concept of spread has been introduced [7], [8]. Spread substitutes the angle notion and hence eliminates the necessity of using transcendental functions as $\sin$ and $\cos$. These ideas has risen a debate about whether the real numbers are necessary or not. No matter how this debate ends, in this paper it is shown that the spread concept has practical implications. The spread concept is extended to perform rotation transforms more efficiently than the standard method. As a consequence, the time for these transformations are significantly reduced. Moreover, the proposed method is easier to program and requires less memory.

The paper is organized as follows. In Section 2 the standard method to perform a rotation transform is revisited. To keep the figures simple the two dimensional case is used. Nevertheless the tree dimensional case is very similar. In Section 3 the spread concept is explained. In this Section is also shown how the spread concept allow to solve trigonometric problems without transcendental functions and why it is necessary to extend the concept to make it useful for coordinates rotation. In Section 4 the spread concept is extended to allow rotation transforms. In Section 5 an analysis is carried out to precise the performance improvement that could be expected with the proposed method in comparison with the standard method. Finally some conclusions are given.

2. The 2d rotation matrix

Let $a = (x, y)$ a point in the coordinate system XY (see Figure 1). Let consider another coordinate system $X'Y'$ with the same origin but rotated an angle α. The coordinates rotation problem is to find the coodinates of $a = (x'y')$ in the new coordinate system.

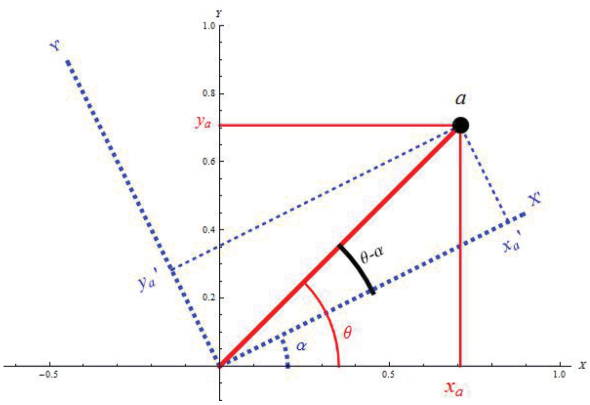

Fig. 1: Coordinates rotation

The usual method to solve this problem is as follows. From Figure 1 note that

$$x = r\cos(\theta) \tag{1a}$$
$$y = r\sin(\theta) \tag{1b}$$

where

$$r = \sqrt{x^2 + y^2}, \quad \tan^{-1}(\theta) = \frac{y}{x} \tag{2}$$

From Figure 1 can also be observed that

$$x' = r\cos(\theta - \alpha) \tag{3a}$$
$$y' = r\sin(\theta - \alpha) \tag{3b}$$

Using trigonometric identities for $\cos(\theta - \alpha)$ and $\sin(\theta - \alpha)$ results

$$x' = r\left(\cos(\theta)\cos(\alpha) + \sin(\theta)\sin(\alpha)\right) \tag{4a}$$
$$y' = r\left(\sin(\theta)\cos(\alpha) - \cos(\theta)\sin(\alpha)\right) \tag{4b}$$

using (1) in (4) yields

$$x' = x\cos(\alpha) + y\sin(\alpha) \tag{5a}$$
$$y' = -x\sin(\alpha) + y\cos(\alpha) \tag{5b}$$

which can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{6}$$

That is, coordinates (x', y') can be obtained multiplying coordinates (x, y) by a Matrix. Such matrix is called the rotation matrix.

In some applications, such as the control of electrical machines, the rotation transform must be performed hundreds of times per second. A $\cos$ and a $\sin$ functions has to be calculated at every time that a rotation transform is performed. Calculation of trigonometric function takes many clock cycles. Hence, a device with high computing power is usually necessary for these applications.

Aimed to reduce the computing power neccesary for some power electronics and electrical machines control applications, in what follows a more efficient method to perform a rotation transform is proposed.

3. Spread and Quadratures

3.1 The spread concept

Consider the right triangle of Figure 2. The "spread" S_1 is defined as

$$S_1 = \frac{y^2}{h^2} \tag{7}$$

where $h^2 = (x^2 + y^2)$. That is squared oposite leg over squared hypotenuse.

Note that the spread S_2 is given by

$$S_2 = \frac{x^2}{h^2} \tag{8}$$

An easy to obtain property but an important one is

$$S_2 = 1 - S_1 \tag{9}$$

It is important to point out that

$$S_1 = \sin^2(\theta) \tag{10}$$

and

$$S_2 = \cos^2(\theta) \tag{11}$$

Note that S_1 and S_2 are always positive and $\sin$ and $\cos$ are squared in (10-11). As it is explained below, this fact prevent the use of spread for coordinates transformation. That is why it is necesary to introduce the extension presented in Section 4

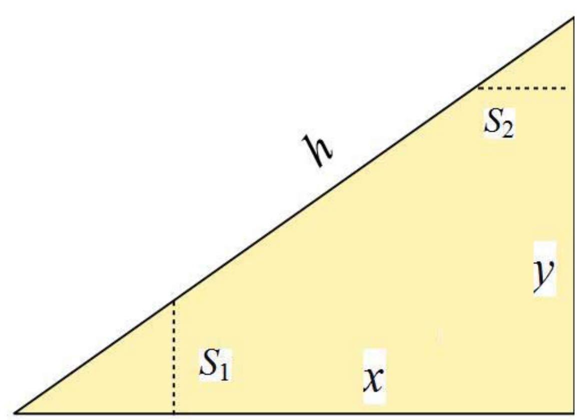

Fig. 2: The spread definition

It can be observed that for a right triangle if a two legs or a leg and spread is known all other data (legs or spread) can be calculated. For example, given x and h (see Figure 2) then S_2 can be calculated from (8). Having S_2, S_1 can be calculated from (9). Finally y can be calculated from (7). Hence, it can be said that it is possible to solve trigonometric problems using the spread concept. Furthermore, trascendental functions are not needed and all the numbers are rational. That is the cause for this trigonometry is called rational.

3.2 The quadrature concept

It is possible to extend the spread concept for not right triangles. Consider three points on a line as it is shown in Fig. 3. Let the distances

$$q_1 = p_2 - p_1, \quad q_2 = p_3 - p_2, \quad q_3 = p_3 - p_1 \tag{12}$$

The cuadrature is defined as the squared distance, that is

$$Q_1 = q_1^2, \quad Q_2 = q_2^2, \quad Q_3 = q_3^2, \tag{13}$$

Since the distance q_1, q_2 and q_3 there is a relation between quadratures. From Fig. 3, it can be observed that

$$Q_3 = Q_1 + Q_2 + 2Q \tag{14}$$

and

$$Q = \sqrt{Q_1}\sqrt{Q_2} \tag{15}$$

Sustituting (15) in (14) yields the so called quadrature equation

$$(Q_3 - Q_1 - Q_2)^2 = 4Q_1Q_2 \tag{16}$$

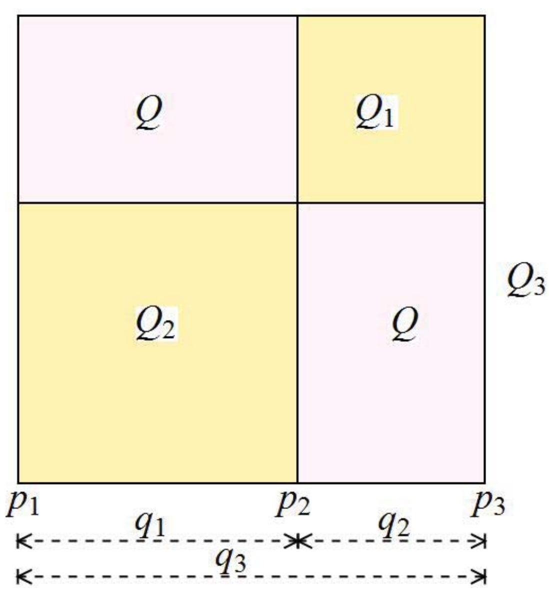

Fig. 3: Quadrature definition.

Now it will be shown that from (16) and the spread concept it is possible to solve trigonometric problems for not right triangles.

First note that any non right triangles can be splitted in two right triangles (see Fig. 4). Using the Pithagoras theorem and the spread, quadratures Q_a, Q_b and Q_h can be expressed as

$$Q_a = Q_1 - Q_h \tag{17a}$$

$$Q_b = Q_3 - Q_h \tag{17b}$$

$$Q_h = S_3Q_1 \tag{17c}$$

As Q_a, Q_b and Q_2 are the quadratures fo three distances in a line, then (16) can be used, yielding

$$(Q_2 + Q_b - Q_a)^2 = 4Q_2Q_b \tag{18}$$

Substituting (17) in (18) results

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 + S_3) \tag{19}$$

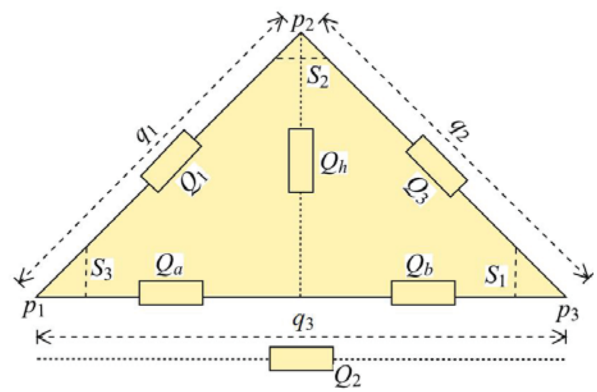

Fig. 4: Quadratures of a triangle

Using (19) it is possible to calculate the spread of two non parallels lines if a point of each line is known. This is a consequence that the intersection point and a point of each line form a triangle (not necessarily a right triangle). The algebra to find the spread of two lines that intersect on the origin is easy and yields

$$S = \frac{(x_2y_1 - x_1y_2)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \tag{20}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of a point in line 1 and line 2 respectively.

3.3 The problem of rotation transform using spread and quadratures

Although expressions (7-9) and (20) allow us to solve trigonometric problems, in its current form they are not useful to perform rotation transforms because two restrictions

- Spread is only defined in the interval $[0 - 1]$ (that is $0 - 90^o$ degrees).

- Spread between two lines does not distinguis which line has more spread (or angle) with respect to X-axis.

These restrictions cause that the four cases depicted in Fig. 5 are indistiguisable in terms of spread because all of them yields the same value. In the next Section the spread concept is equiped with other consideration to distinguish the four cases of Figure 5.

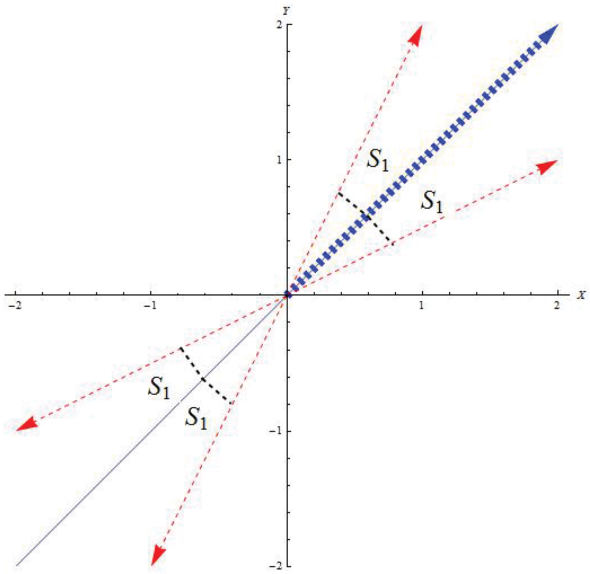

Fig. 5: Four indistinguishable cases

4. Extending the spread concept to allow rotation transforms

In order to distinguish each case of Figure 5 let first distinguish if the vector is in the shaded or non-shaded area of Figure 6.

Consider a point (x_1, y_1) on the positive side of X' axis (see Figure 6). Note that any point (x, y) on the X' axis satifies

$$y = \frac{y_1}{x_1}x \tag{21}$$

or

$$xy_1 - x_1y = 0 \tag{22}$$

From 22, any point on the shaded area satisfies

$$y_1x - x_1y < 0 \tag{23}$$

on the other hand any point on the non-shaded area acomplish

$$y_1x - x_1y > 0 \tag{24}$$

From (10) and (23-24) it can be obtained the following equivalence

$$\sin(\theta) = -\operatorname{sign}(y_1x - x_1y)\sqrt{(S)} \tag{25}$$

Defining $v_1 = -\operatorname{sign}(y_1x - x_1y)$, (25) becomes

$$\sin(\theta) = v_1\sqrt{(S)} \tag{26}$$

Consider now the Figure 7 and a point on the Y' axis. Such a point can be obtained from the point (x_1, y_1) as follows

$$\begin{bmatrix} x_{1\perp} \\ y_{1\perp} \end{bmatrix} = \begin{bmatrix} 0 & (-1) \\ (1) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{27}$$

Having the point $(x_{1\perp}, y_{1\perp})$, it can be seen that any point on the Y' axis satisfies

$$y = \frac{y_{1\perp}}{x_{1\perp}}x \tag{28}$$

or

$$xy_{1\perp} - x_{1\perp}y = 0 \tag{29}$$

From Figure 7 any point (x, y) on the shaded area satisfies

$$y_{1\perp}x - x_{1\perp}y < 0 \tag{30}$$

and the points on the non-shaded area acomplish

$$y_{1\perp}x - x_{1\perp}y > 0 \tag{31}$$

From (11) and (30-31) it can be obtained the equivalence

$$\cos(\theta) = \operatorname{sign}(y_{1\perp}x - x_{1\perp}y)\sqrt{(1 - S)} \tag{32}$$

defining $v_2 = \operatorname{sign}(y_{1\perp}x - x_{1\perp}y)$, (32) becomes

$$\cos(\theta) = v_2\sqrt{(1 - S)} \tag{33}$$

Substituting (26) and (33) in (6) results in

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} v_2\sqrt{1 - S}\sqrt{x^2 + y^2} \\ -v_1\sqrt{S}\sqrt{x^2 + y^2} \end{bmatrix} \tag{34}$$

where S is given by (20)

Note that the rotation matrix given by (34) only requires aritmetic operation, two sign and two square root extraction. It does not require any preprocessing, hence is easier to program and requires less memory.

5. A typical application

To compare the proposed method and the standar procedure a typical engineering problem is considered. Suppose there are two vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$. Consider the problem of decomposing v_2 in two components, one parallel and one orthogonal to v_1 (see Figure 8). Such problem arise in many power electronics applications. The usuall way to proceed is first to calculate the angle θ between the two vectors and then to find the proyection of v_2 on v_1 and its orthogonal vector. In the first step the calculus of a $\tan^{-1}$ function is necesary or a $\sin$ and a $\cos$. The second step requires calculation of at least one $\sin$ and one $\cos$. Because digital systems use power series to evaluate trigonometric functions, a significant amount of time is required to solve this problem. On the other hand only

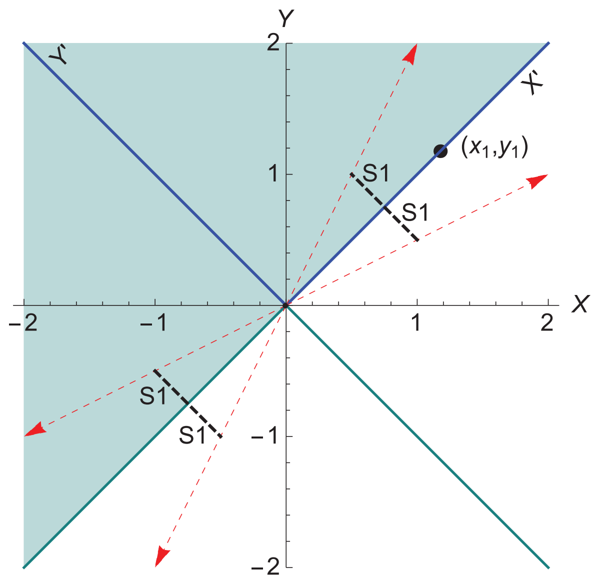

Fig. 6: Establishing the equivalence of sin and S

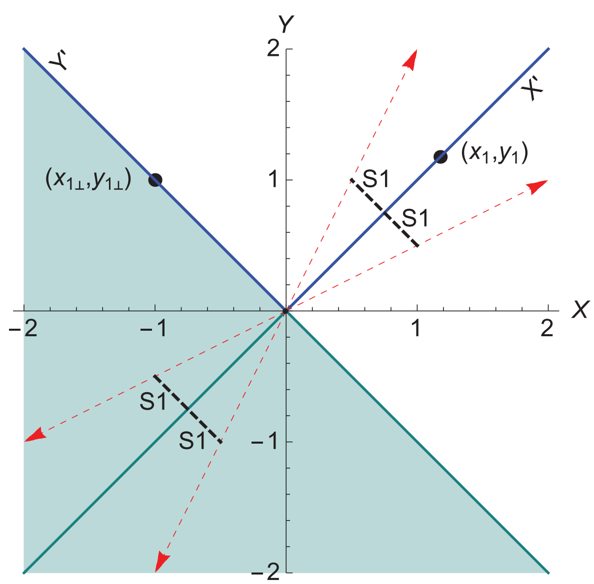

Fig. 7: Establishing the equvalence of cos and S

19 aritmetic operation and two sign operations are needed using the spread concept.

For comparison the two methods were programmed in a Freescale board FRDM-K64F @60MHz using the standard C math library. The test carried out with and without floating point unit (FPU) [9]. Without the FPU the performance of the proposed method was 2.68 times faster than the standar method. When the FPU was used the proposed method is 2.28 faster.

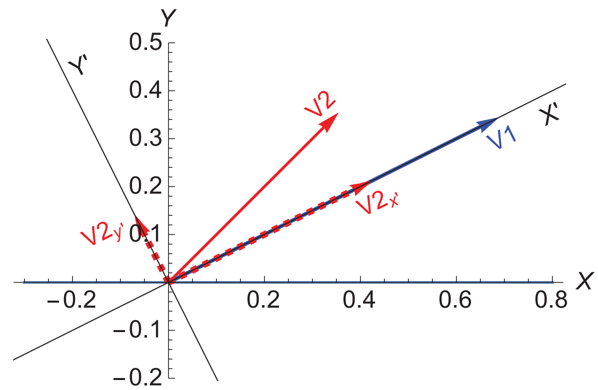

Fig. 8: Typical application of coordinates rotatio

6. Conclusions

Use of spread instead of angles to solve trigonometrical problems has the advantage of only requiring aritmetic oprations. However the spread only works for a quadrant in the plane. In this paper the spread concept has been extended to work in the entire plane. As a result a new rotation matrix that not use sin and cos functions was obtained. The proposed method is faster, easy to program and requires less memory.

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