Accelaration of Poisson Corrupted Image Restoration with Loopy Belief Propagation

Hayaru SHOUNO

1 Graduate School of Informatics and Engineering, University of Electro-Communications,
Chofugaoka 1-5-1, Chofu, JAPAN

Abstract—We treat acceleration of an image restoration throughout Poisson noise channels. Previously, we proposed a image restoration method by use of Expectation Maximization (EM) algorithm[1]. The method requires calculation of inverse of the accuracy matrix, which requires $O(M^3)$ computational cost where $M$ stands for the number of pixels, in order to obtain the posterior mean of some statistics value in each iteration. For reducing the calculation cost, we apply “loopy belief propagation(LBP)” algorithm into our method for the calculation of the marginal posterior means to substitute the posterior mean required in the EM algorithm. As the result, we can accelerate the previous algorithm over 10 times faster in the $80 \times 80$ size image.

Keywords: Poisson image restoration, Latent variational method, Loopy Belief Propagation

1. Introduction

The image restoration problem in the field of digital image processing, is an important in the meaning of the pre-processing of image analysis instrumentation. The Poisson noise corruption process appears in the low contrast object observation such like night photograph, and some kind of computed tomography such like positron emission tomography (PET). We proposed a Poisson noised image restoration in the previous work[1]. In the method, we apply a Bayesian approach to the problem by use of the expectation maximization (EM) algorithm[2][3], which is an iterative type inference algorithm. In the algorithm, we confirmed the success of inference of the hyper-parameter, which controls the strength of the prior knowledge in the Bayesian method, as well as the pixel values. However, our algorithm requires the calculation of the inverse of the accuracy matrix in order to obtain some posterior mean of statistical values. The calculation of the inverse requires $O(M^3)$ computational cost, where $M$ means the total number of the pixels. Thus, our algorithm is hard apply the large scale image restoration.

On the other hand, in the field of image/signal processing, the loopy belief propagation (LBP) algorithm is applied to infer some image restoration problems[4][5][6][7][8]. Even the LBP algorithm is an approximation inference method to obtain some marginal posterior mean, the restoration performance looks good with the appropriate hyper-parameter estimation.

In our previous method, the EM algorithm requires posterior mean. In our new method, we propose the approximation of the posterior mean as the marginal posterior mean. Applying the marginal posterior mean, we might reduce the calculation cost by use of the LBP algorithm. In this study, we investigate the calculation efficacy of the Poisson image restoration by use of the LBP algorithm for the approximation of the posterior mean.

The source code in this paper would be appeared in http://bit.ly/lktgEDM.

2. Formulation

In the formulation, the basic idea is identical to our previous work[1]. Our method is based on the Bayesian approach, so that, at first, we formulate the Bayesian framework, which is consists of image observation process and prior probability.

2.1 Image Observation process

Let us consider $\lambda_i$ as the statistical parameter of the Poisson process where $i$ stands for the pixel position index, and $z_i$ as its random variable, we can describe the observation probability as

$$p(z_i | \lambda_i) = \frac{(\lambda_i)^{z_i}}{z_i!} \exp(-\lambda_i).$$  (1)

Considering the Poisson process, Watanabe et al. treat the corruption process as a Bernoulli process, which counts the number of on-off event in the proper time bins[9]. Thus, we can translate eq.(1) as the binomial distribution form:

$$p(z_i | \rho_i) = \binom{K}{z_i} (\rho_i)^{z_i} (1 - \rho_i)^{K-z_i},$$  (2)
where $\lambda_i = K \rho_i$. In this formulation, we can confirm the eq.(2) converges to the Poisson distribution eq.(1) under the condition $K \to \infty$.

The parameter $\rho_i$ in the eq.(2) is a non-negative parameter, which is just hard to treat for us. Thus, we introduce the logit transform into the parameter $\rho_i$, that is:

$$x_i = \frac{1}{2} \ln \frac{\rho_i}{1 - \rho_i}, \quad (3)$$

and obtain the conditional probability for the condition $x_m$ as

$$p(z_i | x_i) = \left( \frac{K}{z_i} \right) \exp((2z_i - K)x_i - K \ln 2 \cosh x_i). \quad (4)$$

Hence, the image corruption process can be interpreted as observing the $z_i$ under the condition of $x_i$.

### 2.2 Prior probability

Introducing the Bayesian inference requires several prior probability for the image. In this study, we assume some kinds of Gaussian Markov random field (GMRF)[6]. Usually, we define the GMRF as the sum of neighborhood differential square of parameters $\sum_{(i,j)} (x_i - x_j)^2$ where $x_i$ and $x_j$ are neighborhood parameters. The energy function and the prior probability for the GMRF can be described as following:

$$H_{\text{pen}}(\mathbf{x}; \alpha, h) = \frac{\alpha}{2} \sum_{(i,j)} (x_i - x_j)^2 + \frac{h}{2} \sum_{i} x_i^2 \quad (5)$$

$$p(x | \alpha, h) = \frac{1}{Z(\alpha, h)} \exp(-H_{\text{pen}}(\mathbf{x}; \alpha, h)), \quad (6)$$

$$Z(\alpha, h) = \int \mathcal{d}x \exp(-H_{\text{pen}}(\mathbf{x}; \alpha, h)) \quad (7)$$

where $(m, n)$ means the neighborhood pixel indices, and $I$ means the identical matrix. In the eq.(5), the first term means the GMRF part and the second means the Gaussian prior for the zero-center value for stable calculation.

### 2.3 Posterior approximation

Introducing the latent variable approximation for the eq.(4), we can derive the upper limit of the observation process[10][9]. Introducing the variational parameter $\xi_m$, the term $\log 2 \cosh x_m$ can be evaluated as

$$\ln 2 \cosh x \leq \frac{\tanh \xi}{2 \xi} (x^2 - \xi^2) + \ln 2 \cosh \xi. \quad (9)$$

Thus, the observation process can be evaluated as $p(z | x) \geq p_\xi(z | x)$ where

$$p_\xi(z | x) = \prod_i \left( \frac{K}{z_i} \right) \exp \left( \frac{1}{2} x^T \Xi x + z^T x \right) \exp \left( \frac{1}{2} \xi^T \Xi - K \sum_m \ln 2 \cosh \xi_m \right), \quad (10)$$

where $z$ means observation vector

$$z = (2z_1 - K, \ldots, 2z_l - K, \ldots, 2z_M - K)^T. \quad (11)$$

$\xi$ means the collection of latent parameter $\{\xi_m\}$, and matrix $\Xi$ means a diagonal matrix whose components are $\{K \tanh \xi_m \}$. Thus, we approximate the $p_\xi(z | x)$ as the observation process, which is denoted as a Gaussian form.

From the observation (10) and the prior (7), we can derive posterior as

$$p_\xi(x | z, \alpha, h) \propto p_\xi(z | x) p(x | \alpha, h), \quad (12)$$

and the observation can be approximated by the latent-valued form:

$$p_\xi(x | z, \alpha, h) \sim \mathcal{N}(x | \mathbf{m}, (\Xi + \alpha \Lambda + hI)^{-1}), \quad (13)$$

$$\mathbf{m} = (\Xi + \alpha \Lambda + hI)^{-1} z. \quad (14)$$

### 2.4 LBP for corrupted image restoration

In the previous work, we regarded the restoration parameters $x^*$ as the posterior mean

$$x^* = \langle x \rangle = \int \mathcal{d}x \; p_\xi(x | z, \alpha, h) = \mathbf{m}. \quad (15)$$

In order to obtain appropriate restoration, the hyper-parameters $\theta = \{\alpha, h, \xi\}$ should be adjusted properly. Thus, we applied EM algorithm for inferring the hyper-parameters. EM algorithm consists of two-step alternate iterations for the system that has hidden variables. Each time step of EM algorithm indicated by $t$ consists of following two-steps:

- **E-Step**: Calculate Q-function that means the average of the likelihood function for the given parameter $\theta^{(t)}$:

  $$Q(\theta | \theta) = \langle \ln p(x, z | \theta) \rangle_{x \sim | \theta^{(t)}} \quad (16)$$

- **M-Step**: Maximize the Q-function for $\theta$, and the arguments are set to the next hyper-parameters $\theta^{(t+1)}$:

  $$\theta^{(t+1)} = \arg \max_{\theta} Q(\theta | \theta^{(t)}) \quad (17)$$

In each E-step, the inverse of accuracy matrix $(\Xi + \alpha^{(t)} \Lambda + h^{(t)}I)^{-1}$ is required to obtain the hyper-parameters. The computational cost for inverse of a matrix that size is $M \times M$ requires $O(M^3)$ order. Assuming the restoring image size is $L_x \times L_y$, the matrix size becomes $M = L_x L_y$. 

we should integrate the message of the message passing. Here, considering the message as \( j \) we denote the message from the \( j \)th unit to \( i \)th unit as \( M_{j \rightarrow i}(x_i) \). Thus, we introduce the following notations:

\[
M_{j \rightarrow i}(x_i) = \int dx_j p(y_j | x_j) \exp\left(-\frac{\alpha}{2} (x_i - x_j)^2 - \frac{h}{2} x_j^2 \right) \prod_{k \in N(j) \setminus i} M_{k \rightarrow j}(x_j),
\]

where \( N(j) \) means the collection of the connected units to the \( j \)th unit, and \( N(j) \setminus i \) means the collection except \( i \)th unit. From the form of the integral in the eq.(22), we can regard the message from the \( j \)th node to the \( i \)th node as the following Gaussian

\[
M_{j \rightarrow i}(x_i) \propto \mathcal{N}(x_i | \mu_{j \rightarrow i}, \gamma_{j \rightarrow i}^{-1}).
\]  

Substituting the message form eq.(23) into the eq.(22), we can derive the message update rule as

\[
u_{j \rightarrow i} = \frac{\beta_j y_j + \sum_{k \in N(j) \setminus i} \gamma_{k \rightarrow j} M_{k \rightarrow j}(x_j)}{\beta_j + \sum_{k \in N(j) \setminus i} \gamma_{k \rightarrow j} + h}
\]

\[
\frac{1}{\gamma_{j \rightarrow i}} = \frac{1}{\alpha} + \frac{1}{\beta_j + \sum_{k \in N(j) \setminus i} \gamma_{k \rightarrow j} + h}.
\]

The LBP requires iterations for convergence of the message values. After the convergence, the marginal posterior required for the EM algorithm can be evaluated as

\[
p(x_i \mid y, \alpha, h) \propto p(y_i \mid x_i) \prod_{j \in N(i)} M_{j \rightarrow i}(x_i),
\]

\[
p(x_i, x_j \mid y, \alpha, h) \propto p(y_i \mid x_i)p(y_j \mid x_j) \prod_{k \in N(i) \setminus j} M_{k \rightarrow i}(x_i) \prod_{l \in N(j) \setminus i} M_{l \rightarrow j}(x_j).
\]

Thus, the Q-function for the proposing EM algorithm is

\[
Q(\theta \mid \theta^{(t)}) = \langle \ln p(x, y \mid \theta) \rangle_{\text{MP}} = \frac{1}{2} \sum_i \ln \beta_i - \sum_i \frac{\beta_i}{2} \langle (y_i - x_i)^2 \rangle_{\text{MP}} + \frac{M - 1}{2} \ln \alpha - \frac{\alpha}{2} \sum_{(i,j)} \langle (x_i - x_j)^2 \rangle_{\text{MP}},
\]

where \( \langle \cdot \rangle_{\text{MP}} \) means average over the marginal posterior eqs.(26) and (27). Deriving the eq.(28), we assume the hyper-parameter \( h \) is enough small \( h/\alpha \ll 1 \).

Let put them all together, the proposing EM algorithm is shown as the algorithm 1.

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**Fig. 1:** Schematic diagram of message passing of the LBP: The LBP algorithm can be applied to infer the marginal posterior. Each circle shows the pixel, which has 4 nearest neighbors. For instance, considering the message from the \( j \)th unit to \( i \)th unit named \( M_{j \rightarrow i}(x_i) \), the message integrates the messages from the \( j \)th nearest neighbor except \( i \)th.

In order to reduce the cost, we introduce the loopy belief propagation (LBP) in order to infer the parameters. In the manner of the Gaussian graphical model, the efficacy of the LBP were confirmed[6][5]. Our approximated posterior, that is eq.(12), is denoted as a kind of Gaussian form, so that we can apply the LBP for the restoration. For applying LBP, we should change the evaluation of restoration value into the marginal posterior \( p_{y_i}(x_i \mid \alpha, h) \) average:

\[
x^*_i = \langle x_i \rangle = \int dx_i x_i p_{y_i}(x_i \mid \alpha, h).
\]  

Obtaining the marginal posterior mean, we apply a local message passing algorithm defined by LBP. For convenience, we introduce the following notations:

\[
\beta_i = K \frac{\tanh \xi_i}{\xi_i},
\]

\[
y_i = 2z_i - K \frac{\xi_i}{\beta_i}.
\]  

Then we obtain the observation likelihood for \( i \)th node as

\[
p(y_i \mid x_i) \propto \exp\left(-\frac{\beta_i}{2} (y_i - x_i)^2 \right).
\]  

The LBP algorithm is a kind of local message passing. Here, we denote the message from the \( j \)th node to the \( i \)th node as \( M_{j \rightarrow i}(x_i) \). Fig.1 shows the schematic diagram of the message passing. Here, considering the message \( M_{j \rightarrow i}(x_i) \), we should integrate the message of the \( j \)th connected units except \( i \)th. In each LBP iteration, this message passing is carried out for each connection. In the Gaussian MRF case, the message can be derived as
3. Computer Simulation

In order to evaluate the acceleration efficacy, we measured the computational time for restoration both the LBP restoration and the previous algorithms. We adopt the following parameters in the simulations: \( K = 1000, \xi = 10\), and the initial hyper-parameters are \( \alpha^{(0)} = 1.0 \) and \( \xi^{(0)} = 0 \) respectively. The LBP convergence condition is relative error between current and previous state of the pixels \( \left( \sum |m_{i}^{\text{LBPnew}} - m_{i}^{\text{LBPold}}| \right)/ \sum |m_{i}^{\text{LBPold}}| < 10^{-9} \) in the meaning of the LBP iterations. Also the condition for the EM algorithm is set to \( \left( \sum |m_{i}^{\text{EMnew}} - m_{i}^{\text{EMold}}| \right)/ \sum |m_{i}^{\text{EMold}}| < 10^{-5} \). For these conditions, the typical number of convergence for the LBP requires below 100 iterations. And the typical number of iterations for the EM algorithm requires about 200 updates from the initial hyper-parameter state.

Controlling the image noise level of the Poisson corruption, we introduce the linear gray-level transformation from the original pixel values to the range \([2, \text{Max}]\). The larger the maximum value of the range \( \text{Max} \), which is \( 20 \) in this case, controls the Poisson noise corruption level. The middle one shows the Poisson noise corrupted image. The gray pixel means that its pixel value is out of the range of the original image range \([2, 20]\). The right one shows the restored image from the middle one. We can see the restored image looks less impulse noise and smoother rather than the corrupted one.

4. Results

We compare the elapsed times for restoration between the previous our work and proposed one that is the LBP applying methods. Fig.3 shows the result. The horizontal axis shows the image scale of one side \( L_x \), which equals to the other side \( L_y \). Thus, the horizontal axis shows the square root of the total number of image pixels. The vertical one shows the elapsed time for restoring by use of the EM algorithm from the initial hyper-parameter state. Note that the vertical axis is applied the log scale. In the figure, the solid lines show the results for the LBP, and the dashed one show the our previous work called “exact solution”, which solve the inverse of accuracy matrix in each EM step. In the exact solution expressed as the dashed lines, the larger the image size is, the larger elapsed time becomes. We can also see the corruption level does not affect to the calculation cost for the previous work. On the contrary, in the LBP solutions, the calculation cost looks insensitive to the image size. Instead, the LBP solutions are affected to the corruption level, when
Algorithm 1: Poisson corrupted image restoration using EM algorithm with LBP
1: Set the initial hyper-parameters $\alpha^{(0)}$, $\xi^{(0)}$, and $h$
2: Set the initial restoration image $x_{i}^{(0)}$
3: $t \leftarrow 0$
4: repeat
5: Set $\beta_{i}^{(t)} = K \cdot \frac{\tanh\xi^{(t)}}{\beta_{i}^{(t)}}$, and $y_{i}^{(t)} = (2z_{i} - K)/\beta_{i}^{(t)}$.
6: Carry out the LBP, where update eqs. are (24) and (25), under the given hyper-parameters $\alpha^{(t)}$, $\{\beta_{i}^{(t)}\}$.
7: After convergence of the LBP, solve several statistics: the restoration pixel values $\{m_{i}\}$, those of variances $\{(\sigma_{i})^{2}\}$, and the correlations $\{s_{ij}\}$:
\[
\begin{align*}
    m_{i} &= \frac{\beta_{i}^{(t)} y_{i}^{(t)} + \sum_{j \in N(i)} \gamma_{j \rightarrow i} \mu_{j \rightarrow i}}{\beta_{i}^{(t)} + h + \sum_{j \in N(i)} \gamma_{j \rightarrow i}} \\
    (\sigma_{i})^{2} &= \frac{(\beta_{i}^{(t)} + h + \sum_{j \in N(i)} \gamma_{j \rightarrow i})^{-1}}{M - 1} \\
    s_{ij} &= \frac{(\alpha^{(t)} - \gamma_{i \rightarrow j})(\alpha^{(t)} - \gamma_{j \rightarrow i})}{\alpha^{(t)^{2}}}
\end{align*}
\]
8: Update the hyper-parameters:
\[
\begin{align*}
    \xi_{i}^{(t+1)} &= \sqrt{m_{i}^{2} + (\sigma_{i})^{2}} \\
    \frac{1}{\alpha^{(t+1)}} &= \frac{(\sum_{(i,j)} (m_{i} - m_{j})^{2} + (\sigma_{i})^{2} + (\sigma_{j})^{2} - 2s_{ij})}{M - 1}
\end{align*}
\]
9: $t \leftarrow t + 1$
10: until restoration image $\{m_{i}\}$ is converged.
11: $x^{*} \leftarrow m$

it becomes large, which is small $\lambda_{\text{Max}}$, the more calculation cost is required. However, in the large scale image, the LBP solutions has advantage to the exact solutions.

In order to evaluate restoration quantitatively, we introduce the peak signal noise to ratio (PSNR). The PSNR is defined as a kind of similarity between the reference image $q^{*}$ and the test image $q$ as:
\[
\begin{align*}
    \text{PSNR}(q, q^{*}) &= 10 \log_{10} \left( \frac{\max q^{*} - \min q^{*}}{\text{MSE}(q, q^{*})} \right)^{2}, \\
    \text{MSE}(q, q^{*}) &= \frac{1}{M} \sum_{i} (q_{i} - q_{i}^{*})^{2}
\end{align*}
\]
Fig. 4 shows the PSNR between original image $\rho_{m}$ and restored image with inverse logit transform. The horizontal axis shows the maximum number of the Poisson parameter $\lambda_{\text{Max}}$ assigned to the original image. The large $\lambda_{\text{Max}}$ means the original image shows high contrast, so that the noise corruption level is low. On the contrary, the small $\lambda_{\text{Max}}$ means the noise corruption level is high. The evaluation is carried out with 10 times trials and plot with median with quantile deviation. Both the results of the LBP line and the Exact line are overlapped almost all, that is the restoration performance in the meaning of the PSNR are equivalent. In the figure, we can see the slightly improvement in high noise level around $\lambda_{\text{Max}} \sim 10$. On the contrary, in the low noise level around $\lambda_{\text{Max}} \sim 160$, all of the image qualities of LBP, exact, and corrupted one look equivalent. This result means that our restoration approach might not degrade the image quality.

5. Summary & Conclusion
In this study, we propose an acceleration method for the Poisson corrupted image restoration with the LBP. In our image restoration framework that is based on the EM algorithm, the inverse of the accuracy matrix is required. The calculation cost of the inverse required $O(M^{3})$ in general.
Thus, we introduce the LBP to infer the statistics parameters. Our method approximate the Poisson corruption process as a Gaussian form, so that, we can easily derive the LBP update rule. To apply the LBP for the EM algorithm, we have to replace the posterior mean with the marginal posterior mean. Moreover, we should consider the effect of two-body interactions to infer the hyper-parameter $\alpha$. Normally, the LBP only consider the single-body marginal posterior described as eq.(26). Only considering the single-body marginal posterior, the correlation of connected two units, which is denoted as $s_{ij}$ in the eq.(31), becomes 0. This means same effect to the naive mean field approximation. Thus, the single-body marginal posterior occur the underestimation of the parameter of $\alpha$. Avoiding the underestimation, we introduce the two-body marginal posterior described as eq.(27) in the hyper-parameter inference. The correlation $s_{ij}$ update rule is derived as the eq.(31), which only requires the local message, so that the cost for the inference does not increase so much. Solving exact correlation between two units requires considering not only the connected bodies effect but also all the other bodies effect. This is the reason for the requiring the inverse of the accuracy matrix in the EM algorithm. We only consider the two-bodies effect, however, the hyper-parameter inference looks work well, and the restoration performance becomes same or more than the that of the exact solution in the previous work. Hence, we propose the LBP method is a good approximation for our Poisson corrupted image restoration framework.

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References