Computation of Control Related States of Top Left K-net System (with a Nonsharing Resource Place) of Petri Nets

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Abstract - Earlier, Chao pioneered the very first closed-form solution of the number of reachable and other states for marked graphs (MG) and k-th order system which is the simplest class of $S^iPR$ (Systems of Simple Sequential Processes with Resources). This paper progresses one step further on enumerating reachable (forbidden, live and deadlock) states for top Left k-net systems (one non-sharing resource place in the top position of the left-side process, below denoted as Top-Left-k-net) with a formula depending on parameter k for a subclass of nets with k sharing resources.

Keywords: Control systems, discrete event systems, flexible manufacturing systems, Petri nets

1 Introduction

PETRI nets (PN) have been used for modeling and analyzing concurrent systems, such as flexible manufacturing (or resource allocation) systems (FMS) (or RAS) [1-9]. Reachability [11-16] can be used to verify system properties of liveness, boundedness, reversibility, and so on. However, the persistent problem of using PN for modeling various systems is the large number of states generated (called the state explosion problem). It has been shown that the complexity of the reachability problem of a Petri net is EXPSPACE-hard in [12]. Lee et al. [12] show that the reachability problem (whether a marking is reachable) is NP-complete for even a live and safe Free Choice net (LSFC).

To efficiently break the ever exponential time plight of Petri nets, Chao [20, 21, 22] pioneered the very first solution for $S^iPR$ using closed-form methodology by applying simple graph theory and combinatorial mathematics. Here, we extend one step further by constructing a closed form formula of Left k-net systems (Top-Left-k-net) with $r^*$ on the top position of the left-side process.

The approach is explained as follows. The only one token at each $r_i$ initially can stay at $r_o$, $p_i$ (left holder place), or $p'_j$ (right holder place); i.e., 3 possibilities. All together, there are $3^k$ possibilities or states. But some states are not reachable from initial marking via a certain firing sequence. It is easy to find and enumerate the patterns of token distribution for reachable ($\bar{R}$) and live ($L$) states. From that, one can infer the number $F$ of forbidden states as their difference ($F=\bar{R}-L$). The number of nonreachable states is $3^k-\bar{R}$. Reachability problem becomes trivial as one can check whether the marking fits the reachable pattern and hence whether it is reachable.

The rest of the paper is organized as follows. We first list important contributions of our series papers in section II. Based on the results obtained and the methodology of [22], we will then enumerate reachable, forbidden, and live states of Top-Left-k-net system in section III. Finally, Section VI concludes the paper.

2 Contributions of our series papers [20, 22, 24, 25]

Here we define the k-th order system with one non-sharing resource place.

Definition 1: A k-th order system is a subclass of $S^iPR$ with k resource places $r_1$, $r_2$, ..., $r_k$ shared between two processes $N_1$ and $N_2$ and one non-sharing resource place $r^*$ used by an operation place $p^*$ in $P_1$ or $P_2$.

1. $M_0(r^*) = 1$ and $\forall r \in R$, $M_0(r) = 1$.
2. $N_1$ (resp. $N_2$) uses $r_1$, $r_2$, ..., $r_k$ (resp. $r_k$, $r_{k+1}$, ..., $r_{2k}$, $r_1$) in that order.
3. $M_d(p_0) = k+1$, $M_d(p'_0) = k$, where $p_0$ and $p'_0$ are the idle places in processes $N_1$ and $N_2$, respectively.
4. Holder places of $r_j$ in $N_1$ and $N_2$ are denoted as $p_j$ and $p'_j$, respectively.
5. The compound circuit containing $r_o$, $r_{1+1}$, ..., $r_{j-1}$, $r_j$ is called $(r_i-r_j)$-region.
6. If $r'_j$ does not exist, then it is called a k-th order system.
7. There are 3 possibilities for the token initially at $r_i$ to sit at: $p_i(N_1)$, $p'_i(N_2)$, and $r_i$. The corresponding token or $r_i$ state is denoted by $i$, $i$, and 0, respectively.
8. $x^y$ means $r_o$ is at $x$ state ($x=1, 0, -1$) and $r^*$ is at $y$ state ($y=1, 0, -1$), where $h$ is the location of non-sharing resource being used by an operation place $p^*$.

The system is denoted as Top-Left k-th order system when $h=1$ and $p^*$ in $P_1$; Top-Right k-th order system when $h=1$ and $p^*$ in $P_2$; Bottom-Left k-th order system...
when $h = k$ and $p^*$ in $P_2$.

Examples are shown in Figs. 1-5.

### 2.1 K-th order system [20]

Let $N$ be a Petri net; $N'$ is the reverse net of $N$. By the concept of complete reachability graph (Fig. 6) that contains
live, forbidden and nonreachable states, we have Lemma 1, Lemma 2 and Theorem 1.

Lemma 1: Any forbidden state in N is nonreachable in N'.

Lemma 2: Any nonreachable state s in N is a forbidden one or a nonreachable one in N'.

Theorem 1: \( \emptyset(k) = \emptyset(k) - B(k) \), where \( \emptyset(k), V(k), \) and \( B(k) \) are the number of forbidden, nonreachable, and nonreachable + empty-siphon states in a k-th order system, respectively.

In Fig. 5, Deadlock (resp. forbidden but not deadlock, live) states are the nodes that are pointed by a dashed line from D(k). Pointed from V(k) and B(k) are nonreachable states. States nonreachable in both N and N' are (1, 1, 1) and (1 1 1 1). Note that there are no directed paths 1) from a forbidden state to live states, and 2) from reachable states to nonreachable ones.

For the 3rd order system, there are 3 kinds of unmarked (resp. nonreachable) siphon states: (1 1 1), (1 1 1), and (1 0 1) [resp. (1 1 1), (1 1 1), and (1 1 1)], where \( x = 1, 0, 1 \).

Definition 2: \( s = (x_1, x_2, \ldots, x_k) \), \( x_1 = 1, \), 0, 1, \( k \geq 1 \) is a state for a k-th order system N. \( (x_1, x_{i+1}, \ldots, x_{q+i}), k \geq q \geq 1 \) (embedded in s) is a substate of s.

By Definition 2, we have some characteristics of nonreachable and forbidden states of a k-th order system.

A substate of \( (1 1 1) \) \( (x = 1 0 1) \) corresponds to a nonreachable state.

A substate of \( (1 1 1) (x=1 0 1) \) corresponds to a forbidden or a nonreachable state.

State \( s = (x_1, x_2, x_3, x_4, x_5) \) does not carry a substate of \( (1, 1, 1, 1, 1) \) (Fig. 5). For the 3rd order system, there are 3 kinds of unmarked (resp. forbidden or a nonreachable one in N).

Finally, shown below is the total number of each type of states in a k-th order system that we proved in [20].

The total number of states is \( 3^k \).

The total number of live states \( L(k) = 2^{k+1} - 1 \).

The total number of reachable states \( R(k) = (k+2)2^{k-1} \).

The number of forbidden states \( \emptyset(k) = (k-2)2^{k-1} + 1 \).

The number of nonreachable states \( \emptyset(k) = 3^k - (k+2)2^{k-1} \).

The number of nonreachable + empty-siphon states \( B(k) = 3^k - k2^k - 1 \).

Fig. 5 Complete reachability graph of a 3rd-order system (Fig. 1).
The total number of deadlock states $D(k)=k-1$.

2.2 Top-Left, Bottom-Left, and Middle-Left system [22, 24, 25]

**Definition 3:** The equivalent net $N^e = (P^e \cup P_{NR}^e, T^e, F^e)$ of a net $N = (P \cup P_{NR}, T, F)$ (where $P_{NR}$ is the set of non-sharing places) is defined as

1. $P_{NR}^e = P_{NR} \setminus P$;
2. $P^e = P \setminus \bigcup_{r \in P_{NR}} H(r)$;
3. $T^e = T \setminus \bigcup_{r \in P_{NR}} r^*$;
4. $F^e = (F \setminus \bigcup_{r \in P_{NR}} (\langle r, r^* \rangle \cup \langle (r^*), r^* \rangle) \cup \bigcup_{r \in P_{NR}} [(H(r), H(r^*)) \cup (H(r^*), H(r)) \cup \langle (r, r^*), (r^*, r^*) \rangle]$.

**Definition 4:** The reverse net of $N^e$ is denoted as $N^r$.

We say the net in Fig. 3 (k-th order system) is the equivalent of the net in Fig. 5(a), since the Fig. 3 net is exactly the same as the net in Fig. 5(a), except that the nets has one non-sharing resource place $r^*$.

Let $N$ be a net that contains a nonsharing resource in the left process side (for example Top-Left, Bottom-Left, and Middle-Left). Since there are forbidden states in $N^e$ (due to empty-siphon), but live (due to marked siphon) or reachable in $N$ since an empty siphon in $N^r$ may become marked in $N$. We have shown that in $N$ the number of reachable states ($R'$) is $>2R$ and the number of live states ($L'$) $>2L$. To compute $R'$ and $L'$ we need to know how many forbidden and nonreachable states in $N^e$, become reachable or live in $N$.

Because of a nonsharing resource, we have shown that: 1) markings nonreachable in $N^e$, may become reachable in $N$ (denoted the number of which as $\Theta(k)$); 2) forbidden markings in $N^e$ may be live in $N$ (denoted the number of which as $C(k)$); 3) nonreachable markings in $N^r$ may be live in $N$ (denoted the number of which as $A(k)$). We have

$$R' = 2R + \Theta(k).$$

$$L' = 2L + A(k) + C(k).$$

The phenomenon is explained as follows:

$A(k)$: In Bottom-Left structure, by holding at $p^*$ left-side process can wait for right-side process to go through their own work flow. While after firing $t_2$ in Top-Left, it may be live in $N$ [22].

$C(k)$: In Top-Left structure, by holding at $p^*$ left-side process can wait for right-side process to go through their own work flow. While after firing $t_{k+1}$ in Bottom-Left, set of succeeding states belongs to set of unreachable states in $N^e$ [24].

Shown below is methodology of Top-Left [22].

We will derive $R'(\vartheta, L, \ldots etc)$ in terms of $R(\vartheta, L, \ldots etc)$ based on the concept of equivalent k-th order system of a Top-Left k-th order system.

For the 3rd order system, there are 3 kinds of unmarked (resp. nonreachable) siphon states: $(1 -1^1 \ x), \ (x \ 1 -1)$, and $(1 \ 0^1 -1)$ [resp. $(1 \ 1^1 \ x), \ (x \ -1 \ 1)$, and $(1 \ 0^1 \ x)$], where $x=-1, 0, 1$.

A substate of $(1 \ x^1 \ x \ \ldots \ x) (x=1,0,-1)$ in $N^r$ corresponds to a forbidden or a nonreachable state. $L' = 2L + A(k) + C(k) = 18x2^{k-2} + k-4$. 

A substate of $(1 \ x^1 \ x \ \ldots \ x) (x=1,0,-1)$ in $N^r$ corresponds to a forbidden or a nonreachable state.

Let $M$ be a reachable marking in $N'$, then both $M* = M + r*$ and $M' = M + p*$ are reachable in $N$.

Both $s = (1 \ -1^0 \ x_3 \ -x_4 \ \ldots \ x_{j+1} \ x_j)$ and $s' = (1 \ 0^1 \ -1^0 \ x_4 \ \ldots \ x_{j+1} \ x_j)$ where $x_i = 0$, or $-1$, correspond to two legal markings $M$.

Let $M$ be such that only the top $r_{1-r2}$ region in $N^r$ is unmarked.

1) $M$ is nonreachable in $N^r$.
2) $M* = M + r*$ is reachable in $N$.

Let $s = (1 \ 0^0 \ \ldots \ 0_{j+1} \ \ldots \ x_j)$ be such that only the top $r_{1-rj}$ siphon in $N^r$ is unmarked.

1) $M$ is nonreachable in $N^r$.
2) $M* = M + r*$ is reachable in $N$.
3) The total number of such $M*$ is $2^{l_{j-1}}$.

The total number of reachable states in $N$ is $R = 2R + 2^{(k-1)} - (2k + 5)2^{(k-2)} - 1$.

Let $s = (1 \ 0^0 \ \ldots \ 0_{j+1} \ \ldots \ x_j)$ correspond to Marking $M$ such that there are unmarked siphons in only the top $r_{1-r2}$ region in $N^r$. The total number of possible live markings under $M$ is $2^{l_{j-1}}$.

The total number of forbidden markings in $N^e$ that may be live in $N$ is $C(k) = 2^{l_{j-1}} - 1$.

Let $s = (1 \ 1^0 \ \ldots \ x_j)$ correspond to Marking $M$ such that there are unmarked siphons in only the top $r_{1-r2}$ siphon in $N^r$. The total number of possible live markings under $M$ is $1^{l_{j}}$.

The total number of nonreachable markings in $N^e$ that may be live in is $NA(k) = k-1$.
3 Computation of Top-Left-k-net system reachable forbidden and non-reachable states

Definition 7: A k-net system (Top-Left-k-net) is a subclass of S^2PR with k resource places r_1, r_2, ..., r_k shared between two processes N_1, N_2 and one non-sharing resource place r^*_gen (= r^*) used by an operation place p^* in P_i

1. M_0(r^*_gen) = 1 and \( \forall r \in P_R, M_0(r) = 1 \).
2. N_i (resp. N_2, ..., N_k) uses r_1, r_2, ..., r_k (resp. r_k, r_{k-1}, ..., r_2, r_1) in that order.
3. M_0(p^*_j) = k + 1, M_0(p^*_j) = k, i > 1, where p^*_j and p^*_i are the idle places in processes N_i and N_k, respectively.
4. Holder places of r_i in N_i and N_k are denoted as p_j and p^*_j, respectively.
5. The compound circuit containing r_0, r_{i+1}, ..., r_{j-1}, r_j is called (r_{r-r})-region.
6. If r^*_gen does not exist, then it is called a k-net system.
7. \( x^i \) means \( r_{gen+1}^i \) is at x state (x = 1, 0, -1) and \( r^* \) is at y state (y = 0, -1), where gen = i means the location of non-sharing resource being used by an operation place p^*. The system is denoted as Top-Left-k-net system when gen = 1 and \( r^* \) in P_i.

In k-net and Top Top-Left-k-net, let \( y^i_j \) denote the i-th token state at Process j (> 1), \( y^i_0 = 1 \) means the i-th token is at operation place \( p_i \) of Process j, and not at operation place \( p_i \) of other processes. Hence, \( y^i_0 + y^{i+1}_0 + ... + y^{i+n}_0 = y_i=1 \) and there are \((μ-1)\) possibilities, i.e., exactly one of \( y^i_0, y^{i+1}_0, ... y^{i+n}_0 \) equals -1; the rest are 0. \( y_i=0 \) means the i-th token is at resource place \( r_i \). Thus, \( y_i \leq 0 \).

Chao [20] has constructed the formula of \( L_k \) and \( R_k \) for k-net in theorem 2, and 3, as extracted respectively blow:

**Theorem 2 [20]:** For a k-net with \( μ \) processes, the total number of live states is \( L_k = 2^x + (μ-1)^x \) [20].

**Theorem 3 [20]:** For a k-net with \( μ \) processes, the total number of reachable states is \( R(k) = 2^x + (μ-1)^y(1-x^y)/(1-x), \) where \( x = \mu/2 \) and \( y = 2^{k-1} \) [20].

Here we extend to construct the formula of \( L'_k \) and \( R'_k \) for top k-net based on above results. The presence of the non-sharing resource place increases the number of states by a factor of 2. By formula (1) and (2), we can extend to \( L'_k = 2L_k + A'(k) + C'(k), \) where \( A'(k) \) and \( C'(k) \) are defined below:

**Theorem 4:** For a k-net with \( μ \) processes,

1. The total number of forbidden markings that may be live, \( C'(k) = (μ)^{k-1} \).
2. The total number of nonreachable markings that may be live, \( A'(k) = (μ-1)(k-1) \).

**Theorem 5:** For a k-net with \( μ \) processes the total number of live markings \( L'_k = 2L_k + (μ-1)(k-1) + (μ)^{k-1} \)

**Theorem 6:** For a k-net with \( μ \) processes the total number of reachable markings \( R'_k = 2R_k + ((μ)^{k-1} - 1) \).

Examples: Table 3 lists results when \( k = 4, \mu = 3 \) and \( k = 4, \mu = 4 \). These results have been validated experimentally.

| \( k \) | 4 | 4 |
| \( \mu \) | 3 | 4 |

| \( L \) | 146 | 376 |
| \( F \) | 96 | 271 |
| \( D \) | 50 | 105 |
| Total states | 256 | 625 |

| \( C \) | 6 | 9 |
| \( A \) | 26 | 64 |
| \( R \) | 318 | 815 |
| \( L \) | 224 | 614 |
| Total states | 512 | 1250 |

4 Conclusions

We report the very first method to compute in closed form the number of reachable states of Top-Left-k-net system without constructing a reachability graph. This helps to estimate the percentage of deadlocks and legal-state losses due to the addition of a monitor, and avoid the dire situation of mid-run abortion of reachability analysis due to exhausted memory. The formal result is important even if specific since manufacturing systems correspond generally to higher order (k much larger than 3) S^2PR systems, which can model concurrent programs where a locked data item can be represented by a single resource place with one token [27].

Current tools may not be able to handle such high order S^2PR systems due to the state explosion problem.

5 References


