An environment to promote a visual learning of Calculus

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Abstract - *This paper presents a didactical experience taking place at 2012 in a Calculus I course with 24 college students. The scenario, including the simulation of uniformly accelerated motion over a straight line, is analyzed through a dynamical software. Additional to the simulation, SimCalc educational software allows the graphical interplay between velocity and position graphs, and through this visual perception, the emergence of prediction questions about the motion are promoted. Students should deal with them making use of algebraic and numeric procedures related to linear and quadratic equations. Results make us believe in the potential for the learning of calculus of a new way to deal with the visual perception of graphs, as tools for the mathematical reasoning involving numerical, algebraic and graphical representations of functions involved with the visualization of the motion scenario.*

Keywords: Calculus, learning, technology, innovation.

1 Background

As part of a research group working at an educational institution in northern Mexico, our claim since 2000 has been to offer students an innovative approach to the teaching and learning of calculus. We focus on the learning of mathematical knowledge favoring the practical application students will find in later engineering courses.

Based on research we decided to declare the 'practice of predicting the value of a magnitude that is changing' as a 'conducting thread' to reorganize the calculus curriculum. Our approach integrates the ideas of rate of change and accumulation in order to predict this value.

We propose studying motion on a straight line as the first scenario where the magnitudes position and time appear related. We decided this because of Newton's origin of calculus and the familiarity a student could have when thinking about this phenomenon. Through this scenario, students acquire a first impression of calculus concepts and procedures.

Traditional curricula separate differential and integral calculus courses because of the logical structure that traditional textbooks handle. But the reorganization we have been addressing through the prediction practice unifies both courses. This approach to calculus has been discussed and shared in several forums, as well as in educational journals, particularly in [1] and [2].

Through the publication of textbooks and their use in our institution, reflection on the way it impacts the classroom dynamic is our daily source for research in order to strengthen our work. Engineering programs offer the first calculus course with "Applied Calculus: Mathematical competencies through contexts" [3].

Within this innovative approach, we have been working with different software and analyzing its contributions to visual learning. Particular features in technology products do matter for the learning proposal to be strengthened. We find in technology the opportunity to create a way to interact with different representations of mathematical knowledge and carry it into the classroom. We act in accordance with the educational trend [4] named about the role and use of technology: to support an experiential approach to learning.

SimCalc MathWorlds (http://www.kaputcenter.umassd.edu/products/software/), hereon called 'SimCalc', offers dynamically manipulable Cartesian graphs that are linked to motion simulations. Since its conception, James Kaput (1942-2005) was addressing the semiotic perspective of mathematical notations for education; he provided us with a framework for understanding issues that emerged when we attend to the way mathematics comes to be known through the act of teaching and learning within classrooms. SimCalc projects share the common goal to giving students the opportunity to access the important mathematical ideas of change and variation.

SimCalc allows designing of activities in which the goal can vary according to the purpose of the researcher. Particularly, we are concerned on a didactical sequence of activities to create a learning environment in classroom to develop the graphic visualization of the behavior of a function through its derivative. In this paper we want to share the results of the study of a pilot activity regarding the presence of SimCalc in a Calculus I course during spring 2012.

2 Rationale

You can easily recognize in calculus' students the cognitive difficulty [5] identifies dealing with the flexibility of representations. Going back and forth between visual and analytic representations is a matter widely recognized at the core of understanding mathematics. [6] clarifies the notion of representation with underlying cognitive aspects that may be sources of difficulty in learning mathematics. Representations are also signs associated in complex ways, which are produced in compliance with rules to form a system. Like language, they are tools to produce new knowledge as the result of operations and the organization of cognitive structures in our mind.

[7] provides a key insight to analyze the cognitive processes involved in mathematical thinking: you have several systems of representation (algebraic, numeric and graphical) that must be coordinated during mathematical activity. Duval identifies two types of transformations that can be done on those semiotic representations, named as treatment and conversion. The first transformation (treatment) refers to the changes made within the same representation, while the second (conversion) refers to the ability to change representation, including transformation of linguistic statements into numeric, algebraic or graphical.

It is important for calculus students to have a fluent, coordinated and simultaneous use of algebraic symbols, graphics, and graphical and numerical patterns. Currently software is a powerful tool to ‘instantly’ show different representations requested. But even when "the most obvious action is to show several possible representations at the same time" [8] (p. 11), the goal of an appropriate manage of mathematical representations is not immediately attainable.

[8] has recognized different levels for cognitive processes ranging from a recognition process by association that works at a superficial level of the semiotic representation, to a cognitive process that demands some kind of double discrimination acting on different semiotic representations. Activities relying in a mere juxtaposition of multiple simultaneous representations of the same mathematical object promote cognitive processes of the first level, bringing limitations for the learning process.

It is natural to describe a symbol as something that takes the place of another thing. [9] analyzes evolution of symbolic thinking over time. They state that a system of external symbols could become a meta-cognitive mirror in the sense that one's ideas about some field of knowledge can be socially shared; “then we can see our own thought reflected in that "system" and discover something new about our own thinking” [9] (p. 101).

Using computers could transform mathematical knowledge. [10] has been working on this matter looking for new theoretical and methodological tools to enlighten the learning process related to technology integration. By means of a situated abstraction, [10] refers to a way in which a community of students can develop a common discourse and coincide with their teacher about how they are talking of the same mathematical abstractions. It could happen that different meanings are associated with the abstraction, but this benefits the learning situation because differences are explicit to reflect and discuss, and to promote them becoming associated with the teacher’s ones. This way of thinking makes the student’s expressions gain some mathematical legitimacy “even if they differ substantially from traditional mathematical discourse” [10] (p. 2).

[11] addresses the issue of conceiving new ways to think on students’ meanings. Students are so familiar with computational tools, that eventually, the exploration allows them to reorganize strategies for problem solving. Through an exploring experience they can make some situated remarks within the computer environment. Those observations may relate to a certain property, theorem or formula, where the environment facilitated the identification. This comprises a situated proof, the result of a systematic exploration purposely exploited inside a computational environment in order to “prove” mathematical relationships.

The continuous dynamic media offers a special feature that allows the promotion of co-action between the student and the tool. [12] refers to this concept describing, “how the user of a dynamic environment guides the action upon environment and is guided by the environment as a fluid activity” (p. 505). This dual process of guiding and being guided, points out the intentional attitude when carrying out a human activity.

Duval’s framework gives a convenient way to analyze the introduction of technology in classroom. According to our calculus course, we ask technology to foster the possibility to relate graphical information and language used by teacher and students when dealing with a motion situation. We opt to focus on make students familiar with the interpretation of graphs. This goal certainly needs time to be under cognitive control, but we found it relevant to accomplish, since the other semiotic representations (numeric and algebraic) could be considered in order to articulate thoughts and enrich the cognitive process.

3 The activity designed in SimCalc

Favoring graphics in calculus should promote a visual learning. This intention is fulfilled using SimCalc software that brings mathematical expressivity in the classroom through a simulation where a character performs a motion animation according to certain conditions. Through a SimCalc document we can design a visual scenario on the computer screen to support the cognitive process that takes place when a didactical activity is set. There we can handle the position-time graph and the velocity-time graph in different coordinate systems. At the same time, we can visualize the motion animation that is being performed by the character, and bring to the classroom the prediction of character’s position as the magnitude to analyze.

The experience we share here took place in 2012 spring semester during Calculus I course in charge of one of the authors. The 24 students participated bringing to class their own laptop where SimCalc software was installed. The teacher made the introduction of SimCalc before, when studying uniform motion. There, teacher interacts with the classroom computer, and the group visualizes the display in front to discuss what the character (named Ryan) should do. In a one-hour session the scenario designed with motion at a constant velocity was analyzed; character moves over the horizontal straight line according to the information that velocity and position graphs describe visually. Figure 1 shows the scenario where co-action, situated abstraction and proof, treatment and conversion take place, bringing the opportunity to analyze the learning process emerging. The interaction with SimCalc is dynamical; it is made through the hotspots, black dots over the graphs. Dragging the hotspots allow changes in velocity graph having in response an action over position graph according to the Calculus Fundamental Theorem, part of the infrastructure of the software.

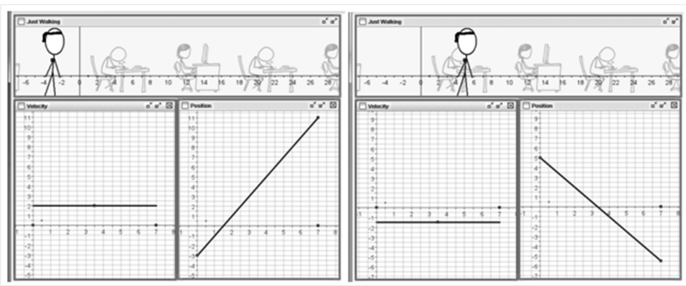

Fig. 1. Two SimCalc documents showing a uniform motion with constant velocity.

The algebraic representation of uniform motion is generalized in class through the identification of several examples with particular numbers involved. The teacher proposes the examples through SimCalc, the specific numeric values are visualized by the graphs (initial values and slopes). The scenario invites students to relate the constant function

$$v(t) = v_0 \tag{1}$$

with the linear function

$$x(t) = x_0 + v_0 t, \tag{2}$$

algebraically and visually. There is identified also the algebraic process of calculating the derivative (from x(t) to v(t)) or the antiderivative (from v(t) to x(t)) between the two functions.

After having studied uniform motion, the textbook provides a numeric strategy and motion arguments in order to relate algebraically the linear velocity function

$$v(t) = v_0 + a_0 t \tag{3}$$

with the quadratic position function

$$x(t) = x_0 + v_0 t + (1/2)a_0 t^2 \qquad (4).$$

Velocity function is identified as the derivative of position, and position function is identified as the antiderivative of velocity. After this, SimCalc brings the opportunity for the graphical interplay with uniformly accelerated motion, the activity we like to share here.

The activity Ryan MUA (*Movimiento Uniformemente Acelerado*) takes place in one-hour class having 2 parts. First, students access the Ryan MUA SimCalc document through the technological platform where the course is distributed. The document shows the same design as in uniform motion, where we privilege showing velocity and position graphs, and the animation of motion performed in a horizontal line. Figure 2 shows two images for linear velocity and the corresponding position graphs.

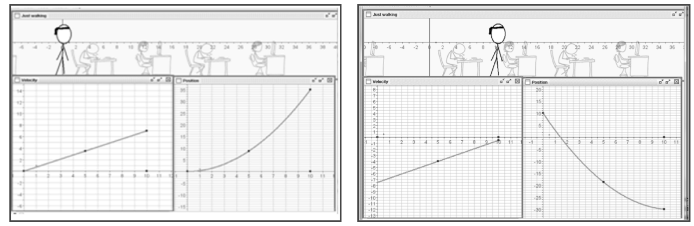

Fig. 2. SimCalc document performing a motion with linear velocity.

The hot spot on the middle of the velocity segment in Figure 2 allows the dragging of velocity segment up and down, and the hotspot at the end of velocity segment allows the dragging for the change of slope. Through the exploration, students get to know about the behavior of position graphs in terms of the behavior of linear velocity. They are encouraged to interact with SimCalc and between each other, and simulate four kinds of motions: to the right increasingly faster (or slower), and to the left increasingly faster (or slower). After half an hour, we propose an image at the screen in front; it is a velocity graph, a linear segment with initial value 5 meters per second on the vertical axis, and intersecting horizontal axis in time 5/2 seconds. We encourage students to interact with Ryan MUA SimCalc document and construct a motion simulation that suits the velocity given. They have to consider that Ryan´s initial position is 1 meter. Activity finishes when they deliver a word document with their picture (screenshot) of SimCalc, the algebraic representation of the motion, and with the interpretation, the later including numeric values of the animation.

4 Results of Ryan MUA Activity in SimCalc

The activity in classroom was done in pairs. Figure 3 shows one of the documents delivered; the description of the animation show that some algebraic procedures were done by hand in order to give numeric details of the motion.

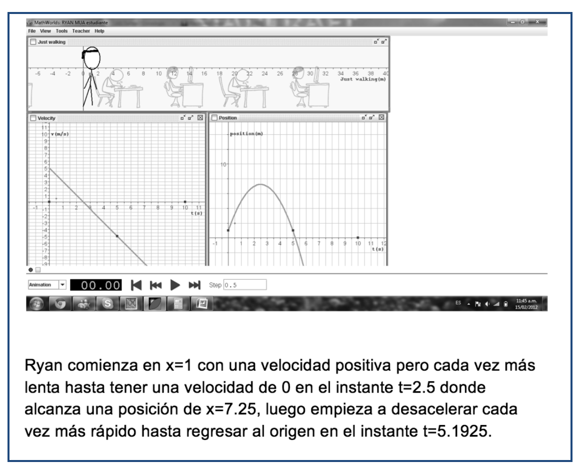

Fig. 3. A students' document with Ryan MUA activity.

Situated in SimCalc scenario, students visualize velocity graph intersecting the time axis, there velocity is zero, and changes from positive to negative values. Corresponding, there is the graphical event for a maximum in position graph. This is the kind of situated proof we have been managing for this calculus result. Class finishes but we ask students for homework to state the situation when velocity graph has initial value -5 and still intersects time axis in 5/2. No change is considered in Ryan's initial position. Homework is delivered by the platform.

Our pedagogical concern in this activity is to get to know the potential of SimCalc to promote the use of the different mathematical representations in order to have a complete description of the motion. It is possible that the visual image of position graph could work like a "meta-cognitive mirror" asking for the mathematical way to catch the time when Ryan crosses the origin (position equals 0 meters) of the straight line where motion takes place. This carries to the establishment of a quadratic equation and its solution, something to remember from students past experience with algebra.

We analyzed the information obtained; certainly we could not say that the activity is easy to all students, but the excitement we feel from them, makes us explore the way to improve the experience. We state the results organized in the Figure 4 where we tried to articulate the potential achievements when Ryan MUA SimCalc accompanies our thinking.

	Graphical representation		**Algebraic** representations (position and velocity)		
Total	Correct image with initial position 1	Incorrect image because initial position was also changed to -1	Correct notation and procedures	There are wrong procedures	Representations not given
24 students	19	5	12	5	7
100 percentage	79.17%	20.83%	50.00%	20.83%	29.17%

Interpretation of MUA motion			**Procedures** and explanations			**Excellent work**
Detailed and correct description	Correct description but incomplete	Description containing mistakes	There is a clear process of conversion between mathematical representations	There appear some inaccuracies with notations	No procedures, no explanations	**Excellent manage of mathematical representations**
12	6	6	10	5	9	6
50	25	25	41.67	20.83	37.50	25

Fig. 4. Results obtained during the activity.

5 Concluding remarks

Nowadays we can deal with the plotting of functions at a very early stage of the calculus' discourse. Technology software offers the opportunity for visual engagement and interaction. Through the experience we showed with SimCalc in classroom, calculus content seems better organized for students, offering powerful ideas to work with earlier. Motion situations over a straight line symbolized graphically allow students to feel confident with the way they communicate results, knowing what they are talking about when dealing with a function and its derivative.

We experienced how SimCalc could promote the learning through an initial global image that relates phenomenon of change of position in the motion context, and with the visual perception motivated by the interrelated graphs of velocity and position. In this scenario, co-action encourages the immersion to get to know how velocity behavior supports position behavior. This idea is important to understanding calculus as the mathematics of change and variation. That is why our concern since 2012 has been to deepen on it by using the PhD dissertation of one of the authors trying to control the desired events that technologies like SimCalc could bring to the learning of calculus.

We strongly believe that the graphical representation of function and its derivative could offer a new way to interpret globally the behavior of a magnitude that is changing. The relationship between the velocity-time graph and the position-time graph is part of a learning visual environment where the description of the motion is expressed through the behavior of those graphs.

Introducing SimCalc into the classroom has been a strong support to our approach for the learning of calculus, giving the opportunity to the reorganization of curriculum we have been working on. A new way to conceive graphs as common tools for mathematical reasoning brings into the classroom a powerful place for visual learning. Through the activity we discussed in this paper we showed how we can talk visually about maximum and minimum within a first meaning. In this way, the study of algebraic contents have meaning in order to access information about the motion and in order to plot functions, a goal that is desired at the end of a first calculus traditional course.

Now we can change the paradigm, we can deal with the graph of a function as the answer for predicting the behavior of a magnitude that is changing, and this since the beginning of the course. The graphical representation of functions is becoming a new way to think of and deal with change and variation phenomena, and it is the place where visual learners of this millennium will have a great opportunity to support their thoughts. Certainly, life in this millennium is an invitation to be visual.

Perhaps the notion of visualization brings the opportunity to identify the kind of cognitive action supported over a visual representation in order to extract information that could be expressed in numbers, in algebraic expressions or even in words. Visualization implies a consistent and organized reasoning involving all these elements and capable of producing new answers and also questions about the global event that stands in front of our eyes. Our proposal is to focus on graphs of both function and derivative, and promote cognitive actions in them, mathematical actions that will make numeric and algebraic information emerge. This is where our thoughts are looking for reaching a new educational technology paradigm, one that our new generations of learners deserve.

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