Tackling Sequences From Prudent Self-Avoiding Walks

Shanzhen Gao, Keh-Hsun Chen
Department of Computer Science, College of Computing and Informatics
University of North Carolina at Charlotte, Charlotte, NC 28223, USA
Email: sgao3@uncc.edu, chen@uncc.edu

***Abstract*—A self-avoiding walk (SAW) is a sequence of moves on a lattice not visiting the same point more than once. A SAW on the square lattice is prudent if it never takes a step towards a vertex it has already visited. Prudent walks differ from most subclasses of SAWs that have been counted so far in that they can wind around their starting point. Some interesting problems and sequences arising from prudent walks of one-sided and two-sided are discussed in this paper. A few methods such as computational, kernel, generating function, recurrence relation and constructive method are applied to our study. Several open problems are posted.**

Keywords: Self-avoiding walk, prudent self-avoiding walk, generating function, kernel method, integer sequence

I. Introduction

A well-known long standing problem in combinatorics and statistical mechanics is to find the generating function for self-avoiding walks (SAW) on a two-dimensional lattice, enumerated by perimeter. A SAW is a sequence of moves on a square lattice which does not visit the same point more than once. It has been considered by more than one hundred researchers in the pass one hundred years, including George Polya, Tony Guttmann, Laszlo Lovasz, Donald Knuth, Richard Stanley, Doron Zeilberger, Mireille Bousquet-Mélou, Thomas Prellberg, Neal Madras, Gordon Slade, Agnes Dittel, E.J. Janse van Rensburg, Harry Kesten, Stuart G. Whittington, Lincoln Chayes, Iwan Jensen, Arthur T. Benjamin, and others. More than three hundred papers and a few volumes of books were published in this area. A SAW is interesting for simulations because its properties cannot be calculated analytically. Calculating the number of self-avoiding walks is a common computational problem [1], [2], [3].

In order to present our problems and results clearly and efficiently, we introduce some notations in the following.

East step: E or $\rightarrow$ or $(1,0)$, x-step

You can see more in the table below:

$(0,1)$	$(1,0)$	$(1,1)$	$(0,-1)$
$\uparrow$	$\rightarrow$	$\nearrow$	$\downarrow$
N	E	NE	S

$(-1,0)$	$(-1,-1)$	$(-1,1)$	$(1,-1)$
$\leftarrow$	$\swarrow$	$\nwarrow$	$\searrow$
W	SW	NW	SE

$\uparrow^{\geq k}$: k or more than k consecutive $\uparrow$ steps

$\uparrow^{=k}$: k consecutive $\uparrow$ steps

avoiding $\uparrow^{\geq k}$: no k or more than k consecutive $\uparrow$ steps

avoiding $\uparrow^{=k}$: no k consecutive $\uparrow$ steps, but can have more than or less than k consecutive $\uparrow$ steps

$\lfloor x \rfloor$: the largest integer not greater than x, $floor(x)$

$\lceil x \rceil$: is the smallest integer not less than x, $ceiling(x)$

$[x^n]f(x)$ denotes the coefficient of x^n in the power series expansion of a function $f(x)$.

$[x^m y^n]f(x,y)$ denotes the coefficient of $x^m y^n$ in the power series expansion of a function $f(x,y)$.

$\binom{n}{r}$ the number of combinations of n things r at a time.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

In the past few decades, many mathematicians have studied the following two classical problems:

Classical Problem 1

What is the number of SAWs from $(0,0)$ to $(n-1,n-1)$ in an $n \times n$ grid, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Donald Knuth claimed that the number is between 1.3×10^{24} and 1.6×10^{24} for $n = 11$ and he did not believe that he would ever in his lifetime know the exact answer to this problem in 1975. However, after a few years, Richard Schroeppel pointed out that the exact value is $1,568,758,030,464,750,013,214,100 = 2^2 3^2 5^2 31 \times 115\,422\,379 \times 487\,148\,912\,401$ [4], [5], [6]. It is still an unsolved problem for $n > 25$.

Classical Problem 2

What is the number $f(n)$ of n-step SAWs, on the square lattice, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

The number $f(n)$ is known for $n \leq 71$ [4], [5], [7], [8].

It is clear that

$$2^n \leq f(n) \leq 4 \times 3^{n-1}$$

$$f(m+n) \leq f(m)f(n)$$

There exists a constant C such that

$$\lim_{n\to\infty} f(n)^{1/n} = \inf_n [f(n)]^{1/n} = C.$$

$C = 2.64$ (up to 71 steps have been counted).

$C = 2.638$ (up to 91 steps have been counted).

$$f(n) \approx 2.638^n$$

The number of SAWs/ the number of total walks:

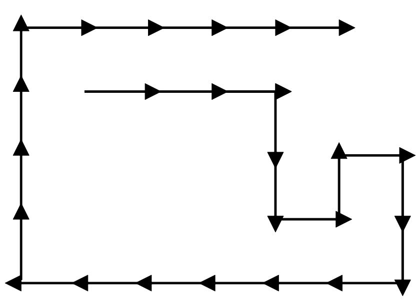
$$\frac{1}{1200} \text{ for } n = 20$$

$$\frac{1}{2.4 \times 10^8} \text{ for } n = 50$$

A recently proposed model called prudent self-avoiding walks (PSAW) was first introduced to the mathematics community in an unpublished manuscript of Préa, who called them *exterior walks*. A prudent walk is a connected path on square lattice such that, at each step, the extension of that step along its current trajectory will never intersect any previously occupied vertex. Such walks are clearly self-avoiding [9], [10], [11], [12], [13]. We will talk about some sequences arising from PSAWs in the following.

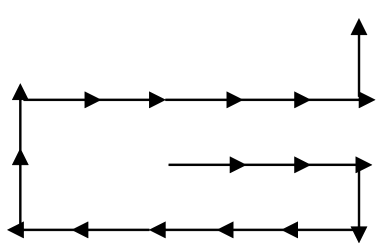
II. Prudent Self-Avoiding Walks: Definitions and Examples

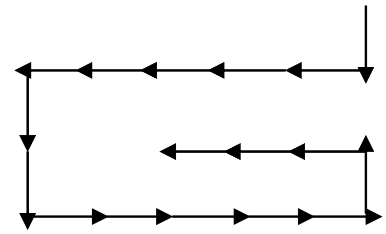
A PSAW is a proper subset of SAWs on the square lattice. The walk starts at $(0, 0)$, and the empty walk is a PSAW. A PSAW grows by adding a step to the end point of a PSAW such that the extension of this step - by any distance - never intersects the walk. Hence the name prudent. The walk is so careful to be self-avoiding that it refuses to take a single step in any direction where it can see - no matter how far away - an occupied vertex. The following walk is a PSAW.



A. Properties of a PSAW

Unlike SAW, PSAW are usually not reversible. There is such an example in the following figure.

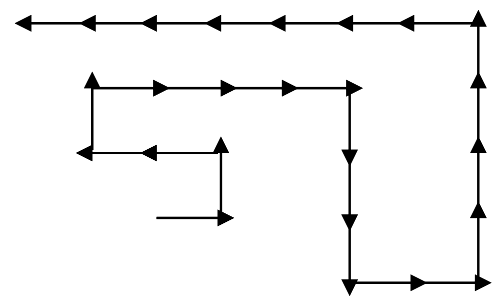




Each PSAW possesses a minimum bounding rectangle, which we call *box*. Less obviously, the endpoint of a prudent walk is always a point on the boundary of the box. Each new step either inflates the box or walks (prudently) along the border. After an inflating step, there are 3 possibilities for a walk to go on. Otherwise, only 2.

In a *one-sided* PSAW, the endpoint lies always on the top side of the box. The walk is *partially directed*.

A prudent walk is *two-sided* if its endpoint lies always on the top side, or on the right side of the box. The walk in the following figure is a two-sided PSAW.



III. Some Sequences Arising from One-sided PSAWs

Sequence 1

What is the number (say $f(n)$) of one-sided n-step prudent walks, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$?

The generating function equals

$$\sum_{n \geq 0} f(n) t^n = \frac{1+t}{1-2t-t^2}$$
$$= 1 + 3t + 7t^2 + 17t^3 + 41t^4 + 99t^5 + ...$$

Also,

$$f(n) = 2f(n-1) + f(n-2)$$
$$= \frac{(1-\sqrt{2})^n + (1+\sqrt{2})^n}{2}$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\sum_{k=0}^{n+1} \binom{n+1}{k} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We obtain sequence $A001333$ of the On-Line Encyclopedia of Integer Sequences.[15, A001333]

Sequence 2

The number of one-sided n-step prudent walks, starting from $(0,0)$ and ending on y-axis, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$ is

$$1+\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \sum_{i=1}^{\min\{n-2k,k\}} \binom{n-2k+1}{i}\binom{k-1}{k-i}\binom{n-k-i}{k}.$$

We obtain sequence $A136029$.[15, A136029]

Sequence 3

Consider the number of one-sided prudent walks starting from $(0,0)$ to (x,y), taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$. The number of such walks with $k+x$ right $\rightarrow$ steps, k left $\leftarrow$ steps and y up $\uparrow$ steps, is

$$\sum_{i=1}^{\min\{y,k+x\}} \binom{y+1}{i}\binom{k+x-1}{k+x-i}\binom{y+k-i}{k}.$$

If $k=2$ and $x=y=n$, we obtain sequence $A119578$.[15, A119578]

Sequence 4

The number of one-sided n-step prudent walks, from $(0,0)$ to (x,y), ($n-x-y$ is even) taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$ is

$$\sum_{i=0}^{\min\{y,\frac{n+x-y}{2}\}} \binom{y+1}{i}\binom{\frac{n+x-y}{2}-1}{\frac{n+x-y}{2}-i}\binom{\frac{n-x+y}{2}-i}{\frac{n-x-y}{2}}.$$

If $x=y=3$, we obtain sequence $A163761$.[15, A163761]

Sequence 5

What is the number of the one-sided n-step prudent walks, avoiding k or more consecutive east steps, $\rightarrow^{\geq k}$?

The generating function equals

$$\frac{1+t-t^k}{1-2t-t^2+t^{k+1}}$$

If $k=1$,

$$\begin{aligned}&\frac{1+t-t^k}{1-2t-t^2+t^{k+1}}\\ &=\frac{1}{1-2t}\\ &=1+2t+4t^2+8t^3+16t^4+32t^5+...\end{aligned}$$

If $k=2$, we obtain sequence $[15, A006356]$:
$1,3,6,14,31,70,157,353,793,1782,4004,8997,20216,$ ...

It also counts the number of paths for a ray of light that enters two layers of glass and then is reflected exactly n times before leaving the layers of glass.

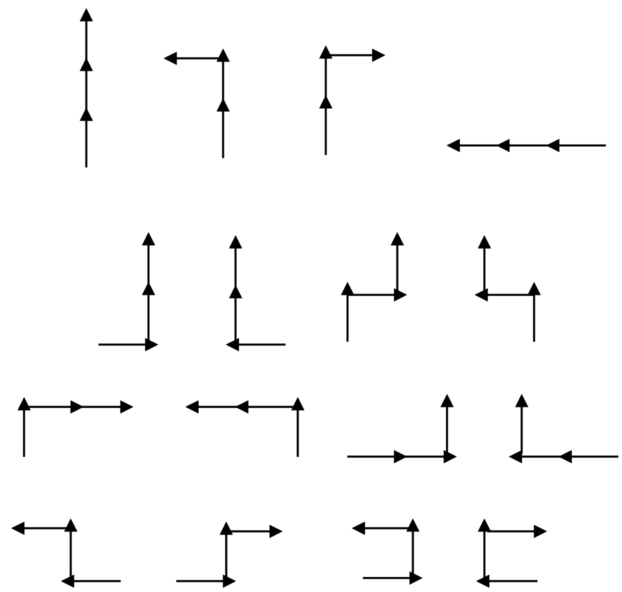
If $k=3$, we obtain sequence [15, A052967]:

$$1,3,7,16,38,89,209,491,1153,2708,6360,...$$

If $k=4$, we obtain sequence [15, A190360]:

$$1,3,7,17,40,96,229,547,1306,3119,3119,7448,...$$

For the case $k=3$ in the above theorem, there are 16 walks as follows:



Sequence 6

The number of one-sided n-step prudent walks, taking steps from $\{\uparrow, \leftarrow, \rightarrow, \nearrow\}$ equals

$$\frac{5+\sqrt{17}}{2\sqrt{17}}\left(\frac{3+\sqrt{17}}{2}\right)^n-\frac{5-\sqrt{17}}{2\sqrt{17}}\left(\frac{3-\sqrt{17}}{2}\right)^n.$$

We obtain sequence $A055099$.[15, A055099]

Sequence 7

What is the number of one-sided n-step prudent walks, taking steps from $\{\rightarrow, \leftarrow, \uparrow, \nearrow, \searrow\}$?

The generating function is

$$\frac{1+t}{1-4t-3t^2}.$$

We obtain sequence $A126473$.[15, A126473]

Sequence 8

What is the number of one-sided n-step prudent walks in the first quadrant, starting from $(0,0)$ and ending on the y-axis, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$?

The generating function is

$$\frac{1}{2t^3}\left((1+t)(1-t)^2-\sqrt{(1-t^4)(1-2t-t^2)}\right).$$

Sequence 9

What is the number of one-sided n-step prudent walks *exactly* avoiding $\leftarrow^{=k}$, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$?

The generating function equals

$$\frac{1+t-t^k+t^{k+1}}{1-2t-t^2+t^{k+1}-t^{k+2}}.$$

If $k=1$, we obtain sequence $A078061$.[15, A078061]

Sequence 10

What is the number of one-sided n-step prudent walks exactly avoiding $\leftarrow^{=k}$ and $\uparrow^{=k}$ (both at the same time)?

The generating function is

$$\frac{1+t-2t^k+2t^{k+1}}{1-2t-t^2+2t^{k+1}-2t^{k+2}}.$$

For $k=1$,

$$f(n)=\left(2^{n+2}-(-1)^{\lfloor n/2\rfloor}+2(-1)^{\lfloor (n+1)/2\rfloor}\right)/5,$$

also,

$$f(n)=2f(n-1)-f(n-2)+2f(n-3)$$
$$\text{with } f(1)=1, f(2)=3, f(3)=7.$$

This is sequence $A007909$.[15, A007909]

IV. Some Sequences Arising from Two-sided PSAWs

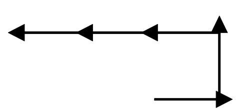
What is the number of two-sided, n-step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$ (both at the same time), taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$?

Theorem 1. *The generating function (say $T(t,u)$) of the above two-sided prudent walks ending on the top side of their box satisfies*

$$\left(1-t^2u-\frac{tu}{u-t}\right)T(t,u)=1+tu+T(t,t)t\frac{u-2t}{u-t}, \quad (1)$$

where u counts the distance between the endpoint and the north-east (NE) corner of the box.

For instance, in the following figure, a walk takes 5 steps, and the distance between the endpoint and the north-east corner is 3. So we can use t^5u^3 to count this walk.



Outline of the proof of the theorem:

Case 1: Neither the top nor the right side has ever moved; the walk is only a west step. This case contributes 1 to the generating function.

Case 2: The last inflating step goes east. This implies that the endpoint of the walk was on the right side of the box before that step. After that east step, the walk has made a sequence of north steps to reach the top side of the box. Observe that, by symmetry, the series $T(t,u)$ also counts walks ending on the right side of the box by the length and the distance between the endpoint and the north-east corner. These two observations give the generating function for this class as $T(t,t)$.

Case 3: The last inflating step goes north. After this step, there is either a west step or a bounded sequence of East steps. This gives the generation function for this class as

$$\left(t^2u+\frac{tu}{u-t}\right)T(t,u)-\frac{t^2}{u-t}T(t,t)$$

Putting the three cases together, we get the generating function (1) for $T(t,u)$.

Solve this generating function for $T(t,u)$ using the Kernel Method:

From

$$\left(1-t^2u-\frac{tu}{u-t}\right)T(t,u)=1+tu+T(t,t)\left(t-\frac{t^2}{u-t}\right),$$

we can get

$$\begin{aligned}&(1-tu)\left(u-tu-t-t^2u^2+t^3u\right)T(t,u)\\&=(u-t)(1-tu)(1+tu)-T(t,t)\,(1-tu)\,t\,(2t-u)\end{aligned}$$

Set $(1-tu)\left(u-tu-t-t^2u^2+t^3u\right)=0$, then there is only one *power series* solution for u

$$u=\frac{1}{2t^2}\left(1-t+t^3-\sqrt{\left(1-t-t^3\right)^2-4t^4}\right).$$

Let U be this solution,

$$U=U(t)=\frac{1}{2t^2}\left(1-t+t^3-\sqrt{\left(1-t-t^3\right)^2-4t^4}\right). \quad (2)$$

Set

$$(1+tu)(u-t)(1-tu)+T(t,t)\,(1-tu)\,t\,(u-2t)=0,$$

and replace u by U:

$$T(t,t)=(1+tU)\,\frac{t-U}{t\,(U-2t)}. \quad (3)$$

From

$$\begin{aligned}&(1-tu)\left(u-t-tu-t^2u^2+t^3u\right)T(t,u)\\&=(u-t)(1-tu)(1+tu)-T(t,t)\,(1-tu)\,t\,(2t-u)\end{aligned}$$

get

$$T(t,u)=\frac{(t-u)(1-tu)(1+tu)}{(1-tu)\left(u-t-tu-t^2u^2+t^3u\right)}+\frac{T(t,t)\,(1-tu)\,t\,(2t-u)}{(1-tu)\left(u-t-tu-t^2u^2+t^3u\right)}$$

Replace $T(t,t)$ by (3). Now

$$\begin{aligned}T(t,u)=&\frac{(1+tu)(u-t)}{u-t-tu-t^2u^2+t^3u}\\&-\frac{(1+tU)\,(U-t)\,(1-tu)\,(u-2t)}{(U-2t)\,(1-tu)\,(u-t-tu-t^2u^2+t^3u)}\end{aligned}$$

where $U(t)$ has been defined in (2).

Sequence 11

Notice that $T(t,1)$ is the generating function of the number of two-sided n-step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, thus $T(t,1)=$

$$\begin{aligned}&\frac{1}{2t\left(1-2t-t^2+t^3\right)\left(1-2t-2t^3\right)}\times\\&\left((1-2t)\,(1-t)\sqrt{\left(1-t-t^3\right)^2-4t^4}-\right.\\&\left.(1+t)\left(1-7t+14t^2-11t^3+10t^4-4t^5\right)\right)\\&=1+3t+6t^2+15t^3+35t^4+83t^5+195t^6+\ldots\end{aligned}$$

Sequence 12

Note that $T(t,0)$ is the generating function of the number of two-sided n-step prudent walks ending at the north-east corner of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, so $T(t,0) =$

$$\frac{(1-t)\sqrt{(1-t-t^3)^2-4t^4}-1+3t-t^2+t^3+t^4}{(1-2t-2t^3)\,t}$$
$$= 1+2t+4t^2+10t^3+24t^4+56t^5+130t^6+304t^7+\ldots$$

Sequence 13

Furthermore, $2T(t,1)-T(t,0)$ is the generating function of the number of two-sided n-step prudent walks ending on the top side or right side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, thus $2T(t,1)-T(t,0) =$

$$\frac{1}{(1-2t-t^2+t^3)(1-2t-2t^3)}\times$$
$$(t(1-t)^2\sqrt{(1-t-t^3)^2-4t^4}+$$
$$1-t-2t^2-2t^3-2t^4+4t^5-t^6)$$
$$= 1+4t+8t^2+20t^3+46t^4+110t^5+260t^6+616t^7\ +\ldots$$

Open Problem 1

What is the number of two-sided n-step prudent walks, ending on the top side of their box, avoiding both $\leftarrow^{\geq k}$, and $\downarrow^{\geq k}$ $(k>2)$ taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$?

The generating function satisfies:

$$\left(1-t^2u\frac{1-t^ku^k}{1-tu}-\frac{tu}{u-t}\right)T(t,u)$$
$$=1+tu\frac{1-t^ku^k}{1-tu}+\frac{u-2t}{u-t}tT(t,t),$$

where u counts the distance between the endpoint and the north-east corner of the box. For $k=3$,

$$\frac{u-t-t^2u^2+t^3u-t^3u^3+t^4u^2-t^4u^4+t^5u^3-tu}{u-t}$$
$$\times T(t,u)$$
$$=1+tu+t^2u^2+t^3u^3+\frac{u-2t}{u-t}tT(t,t)$$

i.e.,

$$(-t+\left(1+t^3-t\right)u+\left(t^4-t^2\right)u^2+\left(t^5-t^3\right)u^3+-t^4u^4)$$
$$\times T(t,u)$$
$$=(1+tu+t^2u^2+t^3u^3)(u-t)+t\,(u-2t)\,T(t,t).$$

Set $-t+\left(1+t^3-t\right)u+\left(t^4-t^2\right)u^2+\left(t^5-t^3\right)u^3-t^4u^4=0$, and solve for u, as a power series of t. We obtained the first one hundred terms for u, beginning with

$$u=t+t^2+t^3+t^4+2t^5+4t^6+8t^7+16t^8+33t^9+69t^{10}+\ldots$$

Using this u, we can get many examples for the sequence.

Open Problem 2

What is the number of two-sided n-step prudent walks, ending on the top side of their box, exactly avoiding both $\leftarrow^{=2}$, $\downarrow^{=2}$, taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$?

The generating function is

$$(1-\frac{t^2u}{1-tu}-\frac{tu}{u-t}+u^2t^3)T(t,u)$$
$$=\frac{1}{1-tu}-u^2t^2+\frac{u-2t}{u-t}tT(t,t).$$

It seems to us it is not trivial to solve this generating function.

V. Some Theorems and Proofs

Theorem 2. *The generating function of the number, say $f(n,k)$, of the one-sided n-step prudent walks, taking steps from $\{\uparrow,\leftarrow,\rightarrow\}$, avoiding k or more consecutive east steps, $\rightarrow^{\geq k}$ satisfies*

$$\frac{1+t-t^k}{1-2t-t^2+t^{k+1}},$$

and for $k\geq 2$,

$$f(n,k)=\sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{\frac{n-j-i}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-j-i}{k-1}}$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{\frac{n-j-i-1}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-i-j-1}{k-1}}$$
$$-\sum_{i=0}^{n-k}\sum_{j=0}^{i}(-1)^{\frac{n-j-i-k}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-j-i-k}{k-1}}$$
$$f(n,1)=2^n.$$

Proof: Let $F(t)$ denote the length generating function of the number of one-sided prudent walks, avoiding k or more consecutive east steps. We have the following three cases.

(1) For the walks which do not contain North steps, they can be empty walk, walks with only west steps, walks with only east steps with length at least one and at most $k-1$, the contributions are 1, $\frac{t}{1-t}$, $\frac{t(1-t^{k-1})}{1-t}$ respectively.

(2) For the walks obtained by concatenating a one-sided walk, a North step, and then a West walk, the contribution is

$$F(t)\frac{t}{1-t}.$$

(3) For the walks obtained by concatenating a one-sided walk, a North step, and then a East walk with at least 1 step and at most $k-1$ steps, the contribution is

$$F(t)\frac{t^2(1-t^{k-1})}{1-t}.$$

Adding these three contributions give the equation

$$F(t)=1+\frac{t}{1-t}+\frac{t(1-t^{k-1})}{1-t}$$
$$+F(t)\frac{t}{1-t}+F(t)\frac{t^2(1-t^{k-1})}{1-t}.$$

Thus,

$$F(t) = \frac{1+t-t^k}{1-2t-t^2+t^{k+1}}.$$

Now, let $[t^n]F(t)$ denote the coefficient of t^n in the power series expansion of $F(t)$.

$$\begin{aligned}
&[t^n]\frac{1+t-t^k}{1-2t-t^2+t^{k+1}} \\
&= [t^n](1+t-t^k)\sum_{i=0}^{\infty}(2t+t^2-t^{k+1})^i \\
&= [t^n](1+t-t^k)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(2t)^{i-j}(t^2-t^{k+1})^j \\
&= [t^n](1+t-t^k) \\
&\times \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(2t)^{i-j}\sum_{l=0}^{j}\binom{j}{l}(t^2)^{j-l}(-1)^l t^{(k+1)l} \\
&= [t^n](1+t-t^k) \\
&\times \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{i}{j}\binom{j}{l}(-1)^{-l}\, t^{i+j-l+lk}2^{i-j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{\frac{n-j-i}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-j-i}{k-1}} \\
&+ \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{\frac{n-j-i-1}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-i-j-1}{k-1}} \\
&- \sum_{i=0}^{n-k}\sum_{j=0}^{i}(-1)^{\frac{n-j-i-k}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-j-i-k}{k-1}}.
\end{aligned}$$

■

Theorem 3. *The number of one-sided n-step prudent walks, starting from $(0,0)$ and ending on the y-axis, taking steps from $\{\uparrow,\leftarrow,\rightarrow\}$ is*

$$1+\sum_{k=1}^{\lfloor (n-1)/2\rfloor}\sum_{i=1}^{\min\{n-2k,k\}}\binom{n-2k+1}{i}\binom{k-1}{k-i}\binom{n-k-i}{k}.$$

Proof: In our proof, we will use the following two results which could be found in some mathematics books such as [16]:

The number of ways of putting n like objects into r different cells is

$$\binom{n+r-1}{n} = \binom{n+r-1}{r-1}.$$

It is also the number of nonnegative integer solutions to the equation

$$\sum_{i=1}^{r} x_i = n.$$

The number of ways of putting n like objects into r different cells with no empty cell is

$$\binom{n-1}{r-1}.$$

It is also the number of positive integer solutions to the equation

$$\sum_{i=1}^{r} x_i = n.$$

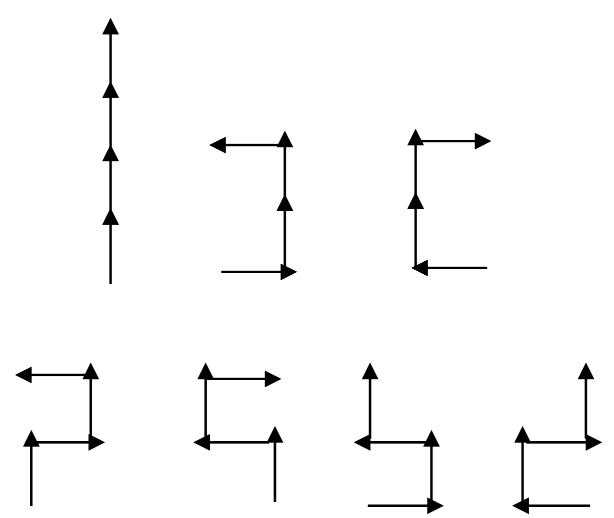
Without loss of generality, we assume that there are k East steps, k West steps and $n-2k$ North steps in a one-sided n-step prudent walks, starting from $(0,0)$ and ending on the y-axis. We also assume that $k>0$ since there is only one such walk for $k=0$. It is easy to see that $k \le \lfloor (n-1)/2\rfloor$. The $n-2k$ North steps provide $n-2k+1$ positions (we can say $n-2k+1$ different cells) for k East steps and k West steps to be inserted. Suppose that we put k East steps into i ($1 \le i \le \min\{n-2k,k\}$) cells with no empty cell. Then there are $\binom{k-1}{k-1}$ ways of putting k East steps into i cells and $\binom{n-2k+1}{i}$ ways of choosing i cells. Now we distribute k West steps into the remaining $n-2k+1-i$ cells, which give us $\binom{n-k-i}{k}$.

Therefore, we get the number:

$$1+\sum_{k=1}^{\lfloor (n-1)/2\rfloor}\sum_{i=1}^{\min\{n-2k,k\}}\binom{n-2k+1}{i}\binom{k-1}{k-i}\binom{n-k-i}{k}.$$

■

Example: For $n=4$ in the above theorem, we have 7 such walks as follows:



Theorem 4. *The number, say $f(n)$, of generalized one-sided n-step prudent walks, taking steps from $\{\uparrow,\leftarrow,\rightarrow,\nearrow\}$ equals*

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{i}{n-i}2^{n-i}(3)^{2i-n} + \sum_{i=0}^{n-1}\binom{i}{n-i-1}2^{n-i-1}(3)^{2i-n+1} \\
&= \frac{5+\sqrt{17}}{2\sqrt{17}}\left(\frac{3+\sqrt{17}}{2}\right)^n - \frac{5-\sqrt{17}}{2\sqrt{17}}\left(\frac{3-\sqrt{17}}{2}\right)^n,
\end{aligned}$$

with generating function

$$\frac{1+t}{1-3t-2t^2}.$$

Proof: Let $P(t)$ denote the length generating function of generalized one-sided prudent walks.

The contribution in $P(t)$ of walks that do not contain North steps or Northeast steps (horizontal walks) is

$$\frac{1+t}{1-t}.$$

The contribution of walks obtained by concatenating a generalized one-sided walk, a North step or Northeast step, then a horizontal walk is

$$\frac{2t(1+t)}{1-t}P(t).$$

Adding these two contributions gives a linear equation for $P(t)$:

$$P(t) = \frac{1+t}{1-t} + \frac{2t(1+t)}{1-t}P(t).$$

Therefore,

$$\begin{aligned} P(t) &= \frac{1+t}{1-3t-2t^2} \\ &= (1+t)\sum_{i=0}^{+\infty}(3t+2t^2)^i \\ &= (1+t)\sum_{i=0}^{+\infty}\binom{i}{j}(3t)^{i-j}(2t^2)^j \\ &= (1+t)\sum_{i=0}^{+\infty}\sum_{j=0}^{i}\binom{i}{j}2^j(3)^{i-j}t^{i+j} \end{aligned}$$

$$\begin{aligned} f(n) &= [t^n]P(t) \\ &= \sum_{i=0}^{n}\binom{i}{n-i}2^{n-i}(3)^{2i-n} \\ &+ \sum_{i=0}^{n-1}\binom{i}{n-i-1}2^{n-i-1}(3)^{2i-n+1}. \end{aligned}$$

The second formula of $f(n)$ can be easily derived from the length generating function. ■

Example: For $n=2$ in the above theorem, we have 14 such walks:

EN,NE, WN,NW,$N(NE)$,$(NE)N$,$E(NE)$,$(NE)E$, $(NE)W$, $W(NE)$, NN, WW, EE, $(NE)(NE)$.

Theorem 5. *The generating function of the number, say $f(n)$, of generalized one-sided n-step prudent walks, taking steps from $\{\rightarrow,\leftarrow,\uparrow,\nearrow,\nwarrow\}$ is*

$$\begin{aligned} &\frac{1+t}{1-4t-3t^2} \\ &= 1+5t+23t^2+107t^3+497t^4+2309t^5 \\ &+ 10\,727t^6+49\,835t^7+\ldots \\ f(n) &= [t^n]\,(1+t)\sum_{k\geq 0}t^k\,(4+3t)^k \\ &= [t^n]\,(1+t)\sum_{k\geq 0}\sum_{m=0}^{k}\binom{k}{m}4^{k-m}3^m t^{m+k} \\ &= \sum_{k=0}^{n}\left[\binom{k+1}{n-k}3+\binom{k}{n-1-k}\right]4^{2k-n}3^{n-1-k}. \end{aligned}$$

Proof: Let $P(t)$ denote the length generating function of generalized one-sided prudent walks.

The contribution in $P(t)$ of walks that do not contain North steps or Northeast steps, or Northwest step (horizontal walks) is

$$\frac{1+t}{1-t}.$$

The contribution of walks obtained by concatenating a generalized one-sided walk, a North step or Northeast step or a Northwest step, then a horizontal walk is

$$\frac{3t(1+t)}{1-t}P(t).$$

Adding these two contributions gives a linear equation for $P(t)$, from which we can get $P(t)$. ■

References

[1] M. Neal and S. Gordon, The Self-Avoiding Walk. Birkhäuser (1996).
[2] L.F. Gregory, Intersections of Random Walks. Birkhäuser (1996).
[3] A.J. Guttmann (Ed.), Polygons, Polyominoes and Polycubes, Springer (2009).
[4] M. Bousquet-Mélou, A.J. Guttmann and I. Jensen, Self-avoiding walks crossing a square, J. Phys. A **38** (2005) 9159–9181.
[5] J. Iwan, Enumeration of self-avoiding walks on the square lattice J. Phys. A **37** (2004) 5503-5524.
[6] K.E. Donald, Science **194** (1976) 1235.
[7] E.J. Janse van Rensburg, Monte Carlo methods for the self-avoiding walk, (Topical Review) J. Phys. A **42** (2009), 323001.
[8] A.R. Conway and A.J. Guttmann, Square lattice self-avoiding walks and corrections to scaling, Phys. Rev. Lett. **77** (1996), 5284–5287.
[9] E. Duchi, On some classes of prudent walks. In Proceedings of the FPSAC'05, Taormina, Italy, (2005).
[10] P. Préa, Exterior self-avoiding walks on the square lattice, (1997) (manuscript).
[11] J.C. Dethridge and A.J. Guttmann, Prudent self-avoiding walks, Entropy **10** (2008), 309–318.
[12] T.M. Garoni, A.J. Guttmann, I. Jensen and J.C. Dethridge, Prudent walks and polygons, J. Phys. A **42** (2009), 095205.
[13] M. Bousquet-Mélou, Families of prudent self-avoiding walks, J. of Combinatorial Theory, Series A **117** (2010) 313–344.
[14] R. P. Stanley, Enumerative Combinatorics I, Wadsworth & Brooks/Cole (1986).
[15] The Online Encyclopedia of Integer Sequences (2014), published electronically at http://oeis.org.
[16] John Riordan, An Introduction to Combinatorial Analysis (originally published: New York: John Wiley 1958), Dover Publications (2002).