Quantum-Union Equivalents of the Orthomodularity Law in Quantum Logic: Part 4

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Abstract

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of quantum-mechanical behaviors and operations. In much the same way that the structure of conventional propositional (Boolean) logic (BL) is the logic of the description of the behavior of classical physical systems and is isomorphic to a Boolean algebra (BA), so also the algebra, C(H), of closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space is a logic of the descriptions of the behavior of quantum mechanical systems and is a model of an ortholattice (OL). An OL can thus be thought of as a kind of “quantum logic” (QL). C(H) is also a model of an orthomodular lattice, which is an OL conjoined with the orthomodularity axiom (OMA). The rationalization of the OMA as a claim proper to physics has proven problematic, motivating the question of whether the OMA and its equivalents are required in an adequate characterization of QL. Here I provide an automated deduction of two quantum-union-based equivalents from orthomodularity theory. The proofs may be novel.

Keywords: automated deduction, quantum computing, orthomodular lattice, Hilbert space

1.0 Introduction

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of the description of quantum-mechanical behaviors ([1], [17], [18], [20]). In much the same way that conventional propositional (Boolean) logic (BL,[12]) is the logical structure of description of the behavior of classical physical systems (e.g. “the measurements of the position and momentum of particle P are commutative” , i.e., can be measured in either other, yielding the same results) and is isomorphic to a Boolean lattice ([10], [11], [19]), so also the algebra, C(H), of the closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space H ([1], [4], [6], [9], [13]) is a logic of the descriptions of the behavior of quantum mechanical systems (e.g., “the measurements of the position and momentum of particle P are not commutative”) and is a model ([10]) of an ortholattice (OL; [4]). An OL can thus be thought of as a kind of “quantum logic” (QL; [19]). C(H) is also a model of (i.e., isomorphic to a set of sentences which hold in) an orthomodular lattice (OML; [4], [7]), which is an OL conjoined with the orthomodularity axiom (OMA; see Figure 1). The rationalization of the OMA as a claim proper to physics has proven problematic ([13], Section 5-6), motivating the question of whether the OMA is required in an adequate characterization of QL. Thus formulated, the question suggests that the OMA and its equivalents are specific to an OML, and that as a consequence, banning the OMA from QL yields a “truer” quantum logic.
Lattice axioms

\[ x = c(c(x)) \]  \hspace{1cm} (AxLat1)
\[ x \lor y = y \lor x \]  \hspace{1cm} (AxLat2)
\[ (x \lor y) \lor z = x \lor (y \lor z) \]  \hspace{1cm} (AxLat3)
\[ (x \land y) \land z = x \land (y \land z) \]  \hspace{1cm} (AxLat4)
\[ x \lor (x \land y) = x \]  \hspace{1cm} (AxLat5)
\[ x \land (x \lor y) = x \]  \hspace{1cm} (AxLat6)

Ortholattice axioms

\[ c(x) \land x = 0 \]  \hspace{1cm} (AxOL1)
\[ c(x) \lor x = 1 \]  \hspace{1cm} (AxOL2)
\[ x \land y = c(c(x) \lor c(y)) \]  \hspace{1cm} (AxOL3)

Orthomodularity axiom

\[ y \lor (c(y) \land (x \lor y)) = x \lor y \]  \hspace{1cm} (OMA)

Definitions of implications and partial order

\[ i_1(x,y) = c(x) \lor (x \land y) \]
\[ i_2(x,y) = i_1(c(y), c(x)) \]
\[ i_3(x,y) = (c(x) \land y) \lor (c(x) \land c(y)) \lor i_1(x,y) \]
\[ i_4(x,y) = i_3(c(y), c(x)) \]
\[ i_5(x,y) = (x \land y) \lor (c(x) \land y) \lor (c(x) \land c(y)) \]
\[ le(x,y) = (x \leq (x \land y)) \]

Definitions of indexed unions

\[ u_1(x,y) = i_1(c(x), y) \]
\[ u_2(x,y) = i_2(c(x), y) \]
\[ u_3(x,y) = i_3(c(x), y) \]
\[ u_4(x,y) = i_4(c(x), y) \]
\[ u_5(x,y) = i_5(c(x), y) \]

where

- \( x, y \) are variables ranging over lattice nodes
- \( \land \) is lattice meet
- \( \lor \) is lattice join
- \( c(x) \) is the orthocomplement of \( x \)
- \( i_1(x,y) \) means \( x \rightarrow_i y \) (Sasaki implication)
- \( i_2(x,y) \) means \( x \rightarrow_2 y \) (Dishkant implication)
- \( i_3(x,y) \) means \( x \rightarrow_3 y \) (Kalmbach implication)
- \( i_4(x,y) \) means \( x \rightarrow_4 y \) (non-tollens implication)
- \( i_5(x,y) \) means \( x \rightarrow_5 y \) (relevance implication)
- \( le(x,y) \) means \( x \leq y \)
- \( \rightarrow \) means if and only if
- \( = \) is equivalence ([12])
- \( 1 \) is the maximum lattice element (\( = x \lor c(x) \))
- \( 0 \) is the minimum lattice element (\( = c(1) \))

**Figure 1.** Lattice, ortholattice, orthomodularity axioms, and some definitions.

There are at least 21 nominal equivalents (in the sense that ortholattice theory, together with these "equivalents", imply the OMA, and vice versa) of the OMA in quantum logic ([5], Theorem 2.5); as nominal equivalents, they are thus of import to optimizing quantum circuit design. Among these is the Proposition shown in Figure 2:

\[ x \cup_i y \rightarrow i c(x) \perp c(y) \]

where
\[ x \perp y \text{ means } \text{le}(x, c(y)) \]
\[ x \cup y \text{ means } c(x) \rightarrow i \text{ } y \]
\[ i = 1, 2, 3, 4, 5 \]

Figure 2. Proposition 2.11 of [5]

2.0 Method

The OML axiomatizations of Megill, Pavićić, and Horner ([5], [14], [15], [16], [21], [22]) were implemented in a prover9 ([2]) script ([3]) configured to derive Proposition 2.11 of [5], for each of \( i = 3, 4, 5 \) from ortholattice theory, then executed in that framework on a Dell Inspiron 545 with an Intel Core2 Quad CPU Q8200 (clocked @ 2.33 GHz) and 8.00 GB RAM, running under the Windows Vista Home Premium /Cygwin operating environment.

3.0 Results

Figure 3 shows the proofs, generated by [3] on the platform described in Section 2.0, that Proposition 2.11 of [5] (for each of \( i = 3, 4, 5 \)) is derivable from orthomodular lattice theory.
\[ \text{le}(x,y) \land \text{c}(\text{c}(x) \lor \text{c}(y)) = x. \]  
\[ \text{perp}(x,y) \land \text{le}(x,\text{c}(y)) \]  
\[ \text{u3}(c_1,c_2) = 1 \land \text{perp}(	ext{c}(	ext{c}_1),\text{c}(\text{c}_2)) \]  
\[ \text{c}(\text{c}_2 \lor \text{c}(\text{c}_1)) \lor (\text{c}_1 \lor (\text{c}(\text{c}_1 \lor \text{c}(\text{c}_2)) \lor \text{c}(\text{c}(\text{c}_1) \lor \text{c}(\text{c}_2)))) = 1 \land \text{perp}(	ext{c}(	ext{c}_1),\text{c}(\text{c}_2)). \]  
\[ \text{c}(1) = 0. \]  
\[ \text{c}(\text{c}(x) \lor \text{c}(x \lor y)) = x. \]  
\[ x \lor \text{c}(\text{c}(x) \lor y) = x. \]  
\[ x \lor y \lor z = (x \land y) \lor z. \]  
\[ x \lor (c(x) \lor y) = 1. \]  
\[ x \lor (c(c(x)) \lor y) = x. \]  
\[ x \lor 0 = x. \]  
\[ x \lor c(y \lor c(x)) = x. \]  
\[ x \lor x = x. \]  
\[ \text{c}(1) = 0. \]  
\[ \text{c}(\text{c}_2 \lor \text{c}(\text{c}_1)) \lor (\text{c}_1 \lor (\text{c}(\text{c}_1 \lor \text{c}(\text{c}_2)) \lor \text{c}(\text{c}(\text{c}_1) \lor \text{c}(\text{c}_2)))) = 1 \land \text{perp}(	ext{c}(	ext{c}_1),\text{c}(\text{c}_2)). \]  
\[ \text{c}(\text{c}_1 \lor \text{c}(\text{c}_2)) = \text{c}(\text{c}_1). \]  
\[ \text{c}(\text{c}_2 \lor \text{c}(\text{c}_1)) \lor (\text{c}_1 \lor (\text{c}(\text{c}_1 \lor \text{c}(\text{c}_2)) \lor \text{c}(\text{c}(\text{c}_1) \lor \text{c}(\text{c}_2)))) = 1 \land \text{le}(\text{c}(\text{c}_1),\text{c}(\text{c}_2)). \]  
\[ \text{c}(1) = 0. \]  
\[ \text{c}(\text{c}(x) \lor \text{c}(x \lor y)) = x. \]  
\[ x \lor \text{c}(\text{c}(x) \lor y) = x. \]  
\[ x \lor y \lor z = (x \land y) \lor z. \]  
\[ x \lor (c(x) \lor y) = 1. \]  
\[ x \lor (c(c(x)) \lor y) = x. \]  
\[ x \lor 0 = x. \]  
\[ x \lor c(y \lor c(x)) = x. \]  
\[ x \lor x = x. \]  
\[ \text{c}(1) = 0. \]  
\[ \text{c}(\text{c}_2 \lor \text{c}(\text{c}_1)) \lor (\text{c}_1 \lor (\text{c}(\text{c}_1 \lor \text{c}(\text{c}_2)) \lor \text{c}(\text{c}(\text{c}_1) \lor \text{c}(\text{c}_2)))) = 1 \land \text{perp}(	ext{c}(	ext{c}_1),\text{c}(\text{c}_2)). \]  
\[ \text{c}(\text{c}_1 \lor \text{c}(\text{c}_2)) = \text{c}(\text{c}_1). \]  
\[ \text{c}(\text{c}_2 \lor \text{c}(\text{c}_1)) \lor (\text{c}_1 \lor (\text{c}(\text{c}_1 \lor \text{c}(\text{c}_2)) \lor \text{c}(\text{c}(\text{c}_1) \lor \text{c}(\text{c}_2)))) = 1 \land \text{le}(\text{c}(\text{c}_1),\text{c}(\text{c}_2)). \]  

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% Proof 1 at 420.06 (+ 7.00) seconds.  
% Length of proof is 56.  
% Level of proof is 11.
12 x v (x ^ y) = x # label("AxL5"). [assumption].
13 x ^ (x v y) = x # label("AxL6"). [assumption].
14 c(x) ^ x = 0 # label("AxOL1"). [assumption].
15 c(x) v x = 1 # label("AxOL2"). [assumption].
16 x v c(x) = 1. [copy(15), rewrite([9(2)])].
17 x ^ y = c(c(x) v c(y)) # label("Df: i4"). [assumption].
27 i4(x,y) = ((c(c(y)) ^ c(x)) v (c(c(y)) ^ c(c(x)))) # label("AxOL3"). [assumption].
28 i4(x,y) = y v (c(y v x) v (c(c(y) v x) v c(c(y) v c(x)))). [copy(27), rewrite([8(3),8(4),8(6),8(6),17(5),17(12),8(11),8(11),9(13),10(13)])].
37 u4(x,y) = i4(c(x),y) # label("Df: u4"). [assumption].
41 -le(x,y) | x ^ y = x # label("Df: less than"). [clausify(1)].
42 -le(x,y) | c(c(x) v c(y)) = x. [copy(41), rewrite([17(2)])].
43 le(x,y) | x ^ y != x # label("Df: less than"). [clausify(1)].
44 le(x,y) | c(c(x) v c(y)) != x. [copy(43), rewrite([17(2)])].
45 -perp(x,y) | le(x,c(y)) # label("Df: perpendicular"). [clausify(2)].
46 perp(x,y) | -le(x,c(y)) # label("Df: perpendicular"). [clausify(2)].
47 u4(c1,c2) = 1 | perp(c(c1),c(c2)) # label("Proposition 2.10u4"). [deny(4)].
48 c2 v (c(c2 v c(c1)) v (c(c1 v c(c2)) v c(c(c1) v c(c2)))) = 1 | perp(c(c1),c(c2)). [copy(47), rewrite([38(3),9(10),9(16)])].
49 u4(c1,c2) != 1 | -perp(c(c1),c(c2)) # label("Proposition 2.10u4"). [deny(4)].
50 c2 v (c(c2 v c(c1)) v (c(c1 v c(c2)) v c(c(c1) v c(c2)))) != 1 | -perp(c(c1),c(c2)). [copy(49), rewrite([38(3),9(10),9(16)])].
51 c1 = 0. [back_rewrite(14), rewrite([17(2),8(2),16(2)])].
52 c(c(x) v c(x v y)) = x. [back_rewrite(13), rewrite([17(2)])].
53 v x v c(c(x) v c(y)) = x. [back_rewrite(12), rewrite([17(1)])].
55 x v (y v z) = y v (x v z). [para(9(a,1),10(a,1,1)), rewrite([10(2)])].
72 c2 v (c(c2 v c(c1)) v (c(c1 v c(c2)) v c(c(c1) v c(c2)))) = 1 | le(c(c1),c(c2)). [resolve(48,b,45,a), rewrite([8(27)])].
78 le(x,y v x). [resolve(52,a,44,b)].
79 c(x) v (c(x v y) = c(x)). [para(52(a,1),8(a,1,1)), flip(a)].
84 c(x v y) v (c(x v y) v x) = c(x). [para(52(a,1),19(a,1,2,1,2)), rewrite([9(5),79(11)])].
90 x v c(c(x) v y) = x. [para(8(a,1),53(a,1,2,1,2))].
94 x v 0 = x. [para(16(a,1),53(a,1,2,1,1)), rewrite([51(2)])].
95 x v c(c(x) v c(y)) = x. [para(19(a,1),53(a,1,2,1,1))].
104 le(x,y v x). [para(9(a,1),78(a,2))].
109 0 v x = x. [para(94(a,1),9(a,1,1)), flip(a)].
219 le(c(c(x) v y),x). [para(90(a,1),104(a,2))].
229 perp(c(x v y),x). [resolve(219,a,46,b), rewrite([8(2)])].
234 perp(c(x v y),y). [para(9(a,1),229(a,1,1))].
290 x v c(c(y v c(x)) v z) = x v z. [para(95(a,1),10(a,1,1)), flip(a)].
371 c2 v (c(c1 v c(c2)) v (c(c2 v c(c1)) v c(c1) v c(c2))) = 1 | c(c1 v c(c2)) = c(c1). [resolve(72,b,42,a), rewrite([8(25)])].
12902 x v (y v (c(z v c(x)) v u)) = y v (x v u). [para(290(a,1),55(a,1,2)), flip(a)].
12922 x v (y v (c(z v c(x)) v u)) = y v (c(z v c(x)) v u) = y v (x v u). [para(84(a,1),290(a,1,2)), flip(a)].
13162 c2 v c(c2 v c(c1)) = 1 | c(c1 v c(c2)) = c(c1). [back_rewrite(371), rewrite([12902,20],95(13),9(7))].
13164 c2 v c(c2 v c(c1)) != 1 | -perp(c(c1),c(c2)).
158756 c2 v c(c2 v c(c1)) = 1 | c(c1 v c(c2)) = c(c1). [para(13162(b,1),8(a,1,1)), rewrite([8(12)])].
160444 c1 v c(c2) = c1. [para(158756(a,1),12922(a,1,2,1)), rewrite([51(9),9(9),10(9)]), flip(b), merge(b)].
160455 perp(c(c1),c(c2)). [para(160444(a,1),234(a,1,1))].
160458 c2 v c(c2) = c1. [para(160444(a,1),95(a,1,2,1))].
161989 $$. [back_unit_del(13164), rewrite([160458,16(4)]), xx(a), unit_del(a,160455)].

Figure 3. Summary of a prover9 (2) derivation of Proposition 2.11, for each of i = 3, 4, 5, from orthomodular lattice theory. The proofs assume the default inference rules of prover9. The general
form of a line in this proof is “line_number conclusion [derivation]”, where line_number is a unique identifier of a line in the proof, and conclusion is the result of applying the prover9 inference rules (such as paramodulation, copying, and rewriting), noted in square brackets (denoting the derivation), to the lines cited in those brackets. Note that some of “logical” proof lines in the above have been transformed to two text lines, with the derivation appearing on a text line following a text line containing the first part of that logical line. The detailed syntax and semantics of these notations can be found in [2]. All prover9 proofs are by default proofs by contradiction.

The total time to produce the proofs in Figure 3 on the platform described in Section 2.0 was approximately 800 seconds.

4.0 Discussion

The results of Section 3.0 motivate several observations:

1. Each of the proofs in Figure 3 uses L1, L2, L3, L5, L6, OL1, OL2, and OL3.

2. The proofs in Section 3.0 may be novel.

3. Companion papers provide proofs for i = 1,2,5 and for Propositions 2.11i, i = 1,2,3,4,5 implies the OMA.

4. Proposition 2.13 can be regarded as a definition of quantum union; thus, this paper together with the papers mentioned in (3), constitute a proof that the definition of quantum intersection is equivalent to the OMA in orthomodular quantum logic. Companion papers derive equivalences for the OMA with definitions of quantum-intersection and quantum-identity. Collectively, these papers provide a theory of equivalence of the OMA with the quantum connectives. In light of these equivalences, QL without the OMA would hardly qualify as a logic.

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6.0 References


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