An Automated Deduction of a "Dishkant-Implication-Restricted" Foulis-Holland Theorem from Orthomodular Quantum Logic: Part 2

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Abstract

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of quantum-mechanical behaviors and operations. The algebra, $C(H)$, of closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space is a logic of the system of "measurement-propositions" quantum mechanical systems and is a model of an ortholattice (OL). An OL can thus be thought of as a kind of "quantum logic" (QL). $C(H)$ is also a model of an orthomodular lattice (OML), which is an ortholattice to which the orthomodular law has been conjoined. An OML can thus be regarded as an orthomodular (quantum) logic (OMLogic). Now a QL can be thought of as a BL in which the distributive law does not hold. Under certain commutativity conditions, a QL does satisfy the distributive law; among the most well known of these relationships are the Foulis-Holland theorems (FHTs). Megill and Pavičić have defined variants of the QL "meet" and "join" connectives in terms of each of the five implication connectives of QL; we can call these variant meet and join connectives "implication-restricted". Here I provide an automated deduction of one of the four Foulis-Holland theorems, restricted to "Dishkant" implication (one of the QL implication-connectives), from OML theory.

Keywords: Dishkant implication, quantum computing, orthomodular lattice, Foulis-Holland theorems

1.0 Introduction

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of the description of quantum-mechanical behaviors ([1], [17], [18], [20]). In much the same way that conventional propositional (Boolean) logic (BL,[12]) is the logical structure of the system of measurement-propositions (e.g., "The position of particle P at time T is X") of classical physical systems and is isomorphic to a Boolean lattice ([10], [11], [19]), so also the algebra, $C(H)$, of the closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space $H$ ([1], [4], [6], [9], [13]) is a logic of the system of measurement-propositions of quantum mechanical systems and is a model ([10]) of an ortholattice (OL; [4]). An OL can thus be thought of as a kind of "quantum logic" (QL; [19]). The lattice and ortholattice axioms, and the Dishkant-implication-restricted variants of "meet" and "join" are shown in Figure 1.
% Lattice axioms
\begin{align*}
x = c(c(x)) & \quad \text{# label("AxL1").} \\
x \lor y = y \lor x & \quad \text{# label("AxL2").} \\
(x \lor y) \lor z = x \lor (y \lor z) & \quad \text{# label("AxL3").} \\
(x \land y) \land z = x \land (y \land z) & \quad \text{# label("AxL4").} \\
x \lor (x \land y) = x & \quad \text{# label("AxL5").} \\
x \land (x \lor y) = x & \quad \text{# label("AxL6").} \\
\end{align*}

% Ortholattice axioms
\begin{align*}
c(x) \land x = 0 & \quad \text{# label("AxOL1").} \\
c(x) \lor x = 1 & \quad \text{# label("AxOL2").} \\
x \land y = c(c(x) \lor c(y)) & \quad \text{# label("AxOL3").} \\
\end{align*}

% Definition of Dishkant implication \((\mathbb{2})\)
\[ i_2(x, y) = c(c(y)) \lor (c(y) \land c(x)) \quad \text{# label("Df: i2").} \]

% Definition of Dishkant-implication-restricted join
\[ u_2(x, y) = i_2(c(x), y) \quad \text{# label("Df: u2").} \]

% Definition of Dishkant-implication-restricted meet
\[ \text{int}_2(x, y) = c(i_2(x, c(y))) \quad \text{# label("Df: int2").} \]

% Definition of \(x\) commutes with \(y\)
\[ C(x, y) \iff (x = ((x \land y) \lor (x \land c(y)))) \quad \text{# label("Df: commutes").} \]

\begin{figure}
Figure 1. Lattice, ortholattice, and ortholattice axioms, and Dishkant-implication-restricted definitions of "meet" and "join". \(x\), \(y\), and \(z\) denote lattice nodes. \(c(x)\) denotes the orthocomplement of \(x\). \(\lor\) denotes lattice join; \(\land\) denotes lattice meet.
\end{figure}

\(C(H)\) is also a model of an orthomodular lattice (OML; [4], [7]), which is an OL conjoined with the orthomodularity axiom (OMLaw):
\[ y \lor (c(y) \land (x \lor y)) = x \lor y \quad \text{(OMLaw)} \]

An OML can thus be thought of an "orthomodular (quantum) logic".

The rationalization of the OMA as a claim proper to physics has proven problematic ([13], Section 5-6), motivating the question of whether the OMA is required in an adequate characterization of QL. Thus formulated, the question suggests that the OMA and its equivalents are specific to an OML, and that as a consequence, banning the OMA from QL yields a "truer" quantum logic.

Now a QL can be thought of as a BL in which the distributive law
\[ (D) \quad (x \lor (y \land z)) = (x \lor y) \land (x \lor z) \]
does not hold. Under certain commutativity conditions, a QL does satisfy \((D)\); among the most well known of these relationships are the Foulis-Holland theorems (FHTs ([7])):
To form an implication-relativized version of any of the Foulis-Holland theorems, I replace meet and join in the consequents of Figure 2 with the corresponding implication-restricted meet and join (e.g., \text{int2} and \text{u2}, as defined in Figure 1, are respectively meet and join defined in terms of implication-connecitive \text{i2} in Figure 1). I denote the resulting Foulis-Holland variant by adding the suffix ".N" to the name of original theorem, where \text{N} = 2 denotes Dishkant-implication-restricted.

\textbf{2.0 Method}

The OML axiomatizations of Megill, Pavičić, and Horner ([5], [14], [15], [16], [21]), were implemented in a \textit{prover9} ([2]) script ([3]) configured to derive \text{FH2}.2, then executed in that framework on a Dell Inspiron 545 with an Intel Core2 Quad CPU Q8200 (clocked @ 2.33 GHz) and 8.00 GB RAM, running under the Windows Vista Home Premium/Cygwin operating environment.

\textbf{3.0 Results}

Figure 3 shows the proof, generated by [3] on the platform described in Section 2.0, that \text{FH2}.2 is a consequence of OML:

\begin{verbatim}
% Foulis-Holland theorem FH1
(C(x,y) & C(x,z)) -> ((x ^ (y v z)) = ((x ^ y) v (x ^ z)) )

% Foulis-Holland theorem FH2
(C(x,y) & C(x,z)) -> ((y ^ (x v z)) = ((y ^ x) v (y ^ z)) )

% Foulis-Holland theorem FH3
(C(x,y) & C(x,z)) -> ((x v (y ^ z)) = ((x v y) ^ (x v z)) )

% Foulis-Holland theorem FH4
(C(x,y) & C(x,z)) -> ((y v (x ^ z)) = ((y v x) ^ (y v z)) )

where C(x,y), "x commutes with y" is defined as
C(x,y) <-> (x = ((x ^ y) v (x ^ c(y))))

\textbf{Figure 2. The Foulis-Holland theorems.}
\end{verbatim}
16 \ ((x \lor y) \land z) = x \lor (y \lor z) \# label("AxL3"). [assumption].
18 \ (x \lor (y \land z)) = x \lor y \# label("AxL5"). [assumption].
19 \ x \land (x \lor y) = x \# label("AxL6"). [assumption].
20 \ c(x) \land x = 0 \# label("AxOL1"). [assumption].
21 \ c(x) \lor x = 1 \# label("AxOL2"). [assumption].
22 \ x \lor c(x) = 1. \{copy(21), rewrite([15(2)])\}.
23 \ x \land y = c(c(x) \lor c(y)) \# label("AxOL3"). [assumption].
27 \ i2(x,y) = c(c(y)) \lor (c(y) \land c(x)) \# label("Df: i2"). [assumption].
28 \ i2(x,y) = y \lor c(y \lor x). \{copy(27), rewrite([14(3),14(3),14(3),14(3)])\}.
37 \ u2(x,y) = i2(c(x),y) \# label("Df: u2"). [assumption].
38 \ u2(x,y) = y \lor c(y \lor c(x)). \{copy(37), rewrite([28(3)])\}.
47 \ int2(x,y) = c(i2(x,c(y))) \# label("Df: int2"). [assumption].
48 \ int2(x,y) = c(c(y) \lor c(c(y) \lor x)). \{copy(47), rewrite([28(3)])\}.
67 \ 1_2 = x \lor ((y \land c(x)) \lor (c(y) \land c(x))) \# label("Df. 2.20"). [assumption].
68 \ x \lor (c(y \lor x) \lor c(c(y \lor x))) = 1_2. \{copy(67), rewrite([14(6),14(6),14(6),14(6),14(6),14(6)])\}.
75 \ x \lor (c(x) \land (y \lor x)) = y \lor x \# label("OMLaw"). [assumption].
76 \ x \lor c(x \lor c(y \lor x)) = y \lor x. \{copy(75), rewrite([23(3),23(10),23(7),23(7),23(10),23(10)])\}.
77 \ int2(x,y) = c(c(x) \lor c(c(x) \lor c(x))) \neq u2(int2(c(x),c(x)),u2(c(x),c(x))) \# answer("Foulis-Holland Theorem 2.2"). [deny(4)].
78 \ c(c(x) \lor c(x \lor c(y \lor x))) \neq c(c(c(x) \lor c(c(x) \lor c(x)))) \lor c(c(x) \lor c(x \lor c(y \lor x))) \# answer("Foulis-Holland Theorem 2.2").
81 \ (c1 \land c3) \lor (c1 \land c3) = c1. \{resolve(12,a,10,a)\}.
82 \ c(c3 \lor c(c1)) \lor c(c3 \lor c(c1)) = c1. \{copy(81), rewrite([23(3),23(10),23(11),23(11),23(11),23(11)])[\}.
83 \ c(0) = 0 \# label("AxOL2"). [assumption].
85 \ x \lor (c(x) \land c(y)) = x. \{back_rewrite(83), rewrite([23(2),23(10),23(7),23(7),23(10),23(10)])\}.
89 \ x \lor (y \lor z) = y \lor (x \lor z). \{para(81), rewrite([22(2),22(2),22(2),22(2),22(2),22(2)])\}.
97 \ x \lor (c(x) \land c(x \lor y)) = 1. \{para(85), rewrite([14(11),14(11),14(11),14(11),14(11),14(11)])\}.
108 \ x \lor (y \lor c(z \lor c(x))) = y \lor x. \{para(89), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
113 \ x = 1_2. \{para(97), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
124 \ x \lor (c(x) \land c(y)) = x. \{back_rewrite(113), rewrite([84(1),84(1),84(1),84(1),84(1),84(1)])\}.
133 \ c(x) \lor c(x \lor y) = c(x). \{para(124), rewrite([14(11),14(11),14(11),14(11),14(11),14(11)])\}.
137 \ c(0) = x. \{para(133), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
140 \ 1 \lor x = x. \{para(137), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
150 \ x \lor 0 = x. \{para(140), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
151 \ x \lor (c(x) \land c(y)) = x. \{para(150), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
158 \ x \lor (y \lor (c(x) \land c(y))) = y \lor x. \{para(151), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
163 \ x \lor (y \lor (c(x) \land c(y))) = y \lor 1. \{para(158), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
193 \ x \lor 1 = x. \{para(163), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
195 \ x \lor (y \lor (c(x) \land c(y))) = 1. \{back_rewrite(193), rewrite([14(3),14(3),14(3),14(3),14(3),14(3)])\}.
207 \ x \lor (y \lor (c(x) \land c(y))) = 1. \{para(195), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
210 \ x \lor (y \lor (c(x) \land c(y))) = 1. \{para(207), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
220 \ c(x) \lor (y \lor (c(x) \land c(y))) = c(x). \{para(210), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
225 \ x \lor (y \lor (c(x) \land c(y))) = x \lor y. \{para(220), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
236 \ c(x) \lor (y \lor (c(x) \land c(y))) = c(x) \lor y. \{para(225), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
247 \ x \lor (y \lor (c(x) \land c(y))) = x \lor c(x). \{para(236), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
260 \ c(x) \lor (y \lor (c(x) \land c(y))) = c(x) \lor y. \{para(247), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
313 \ x \lor (y \lor (c(x) \land c(y))) = x \lor (y \lor (c(x) \land c(y))). \{para(260), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.
370 \ x \lor (y \lor (c(x) \land c(y))) = x \lor (y \lor (c(x) \land c(y))). \{para(313), rewrite([22(1),22(1),22(1),22(1),22(1),22(1)])\}.

3711 x v c(y v c(y v x)) = x. [para(15(a,1),3701(a,1,2,1,2,1))].
3720 x v c(y v x) = x v c(y).
[para(3701(a,1),151(a,1,2,1)),rewrite([15(5),236(5)],flip(a))].
3961 c(c(c2) v c(c3)) v c(c(c1) v (c(c2 v c(c1)) v c(c(c2) v c(c3)))) != c(c(c2) v c(c1 v c(c3))) # answer("Foulis-Holland Theorem 2.2").
[back_rewrite(727),rewrite([3720(8),15(5),3720(21),15(18),3720(35),15(30)])].
4094 x v c(x v y) = x v c(y).
[para(3711(a,1),151(a,1,2,1)),rewrite([15(5),260(5)],flip(a))].
4095 x v (c(y v x) v z) = x v (c(y) v z).
[para(207(a,1),3711(a,1,2,1,2,1)),rewrite([83(6),150(6),15(6),936(6)],flip(a))].
4101 x v (y v c(x v z)) = x v (y v c(z)).
[para(313(a,1),3711(a,1,2,1,2,1)),rewrite([83(6),150(6),15(6),1410(6)],flip(a))].
4104 c(x v y) v c(y v c(x v z)) = c(x).
[para(124(a,1),3711(a,1,2,1,2,1)),rewrite([83(6),150(6),15(6),260(6),15(4),133(4)],flip(a))].
4151 c(c(c2) v c(c1 v c3)) != c(c3 v (c(c1) v c(c2))) v c(c(c2) v c(c3)) # answer("Foulis-Holland Theorem 2.2").
[back_rewrite(3961),rewrite([4095(21),4094(17),14(13),15(12),89(13),15(15)],flip(a))].
4341 x v c(y v (z v x)) = x v c(y v z). [para(16(a,1),3720(a,1,2,1,1))].
4345 c(x) v c(y v c(z v x)) = c(x) v c(y).
[para(3720(a,1),257(a,1,2)),rewrite([257(6)],flip(a))].
4879 x v c(y v c(z v x) v u) = y v (x v c(z v u)).
[para(4095(a,1),89(a,1,2)),flip(a)].
5754 c(x v y) v c(y v c(z v x)) = c(y).
[para(4341(a,1),4345(a,1,2,1)),rewrite([15(11),220(11)])].
5779 c(x v c(y v z)) v c(z v x) = c(x). [para(5754(a,1),15(a,1)),flip(a)]].
5899 c(x v c(y v z)) v c(y v x) = c(x). [para(15(a,1),5779(a,1,1,1,2,1))].
6092 c(x v (y v c(z v u))) v c(z v (x v y)) = c(x v y). [para(16(a,1),5899(a,1,1,1))].
6192 c(x v c(y v z) v c(y v u)) v c(y v (x v c(z))) = (c(x) v c(y v z)).
[para(4101(a,1),5899(a,1,2,1)),rewrite([16(6)])].
6196 c(x v c(y v z) v u) = c(x v z v c(y v u)) v c(x v c(z) v u).
[para(665(a,1),4104(a,1,2,1)),rewrite([14(2),15(5),4871(5),14(12)],flip(a))].
6230 c(x v c(y v z)) v c(x v (z v c(y))) v c(x v c(z)).
[back_rewrite(6192),rewrite([6196(7),133(4),16(16),6092(15)],flip(a))].
6565 $F$ # answer("Foulis-Holland Theorem 2.2").
[back_rewrite(4151),rewrite([6230(8),15(7),16(7)],xx(a))].. 

Figure 3. Summary of a prover9 ([2]) proof of FH2.2 from OML. The proof assumes the default inference rules of prover9. The general form of a line in this proof is “line_number conclusion [derivation]”, where line_number is a unique identifier of a line in the proof, and conclusion is the result of applying the prover9 inference rules (such as paramodulation, copying, and rewriting), noted in square brackets (denoting the derivation), to the lines cited in those brackets. Note that some of “logical” proof lines in the above have been transformed to two text lines, with the derivation appearing on a text line following a text line containing the first part of that logical line. The detailed syntax and semantics of these notations can be found in [2]. All prover9 proofs are by default proofs by contradiction.

The total time to produce the proofs in Figure 3 on the platform described in Section 2.0 was approximately 2.5 seconds.

4.0 Discussion
The results of Section 3.0 motivate several observations:

1. FH2.2 is derivable from OML.
2. The proof in Section 3.0 is, as far as I know, novel.
3. Companion papers show that the other three Dishkant-implication restricted FH theorems are also derivable from OML.
4. The OMLaw is equivalent to each of the four FH theorems ([23]) in OML. Whether a Dishkant-implication-restricted variant of the OMLaw is derivable from OL (i.e., OML without the OMLaw) conjoined with the Dishkant-implication-restricted FH theorems is the subject of future work.

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6.0 References


