The Complexity and Algorithm for k-Duplicates Combinatorial Auctions with Submodular and Subadditive Bidders

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Abstract—In this paper, we study the problem of maximizing welfare in combinatorial auctions with $k(>1)$-duplicates of each item, where $k$ is a fixed constant (i.e. $k$ is not the part of the input) and bidders are submodular or subadditive. We exhibit some upper and lower approximation bounds for $k$-duplicates combinatorial auctions. First, we show that it is NP-hard to approximate the maximum welfare for $k$-duplicates combinatorial auctions with subadditive bidders within a factor of $2-\epsilon$ where $\epsilon > 0$ unless $P = NP$. Secondly, we propose a 2-approximation algorithm for $k$-duplicates combinatorial auctions with submodular bidders.

Keywords: Approximation Algorithm; Combinatorial Auctions; NP-hard

1. Introduction

We consider the allocation problem in combinatorial auctions with $k(>1)$-duplicates of each item where $k$ is a fixed constant. In the past years, there has been much interest in combinatorial auctions. In a combinatorial auction, there are a set $M$ of $m$ items and $n$ bidders. These items are being sold to bidders. Every bidder $i$ has a valuation function (it is also called utility function in some cases) $v_i : 2^M \rightarrow \mathbb{R}^+$. We suppose that the valuation function is monotone, which means for every two bundles $S, T, S \subseteq T \subseteq M$ it holds $v(S) \leq v(T)$, and normalized $v(\emptyset) = 0$. The goal is to find a partition $(S_1, \ldots, S_n)$ of the $m$ items that maximizes the total utility or social welfare, i.e., $\Sigma_i v_i(S_i)$ is maximized. We call such an allocation an optimal allocation.

The $k$-duplicates combinatorial auction is the allocation problem in combinatorial auctions with $k$-duplicates of each item where $k$ is a fixed constant. Every bidder is still interested in at most one unit of each item and every valuation is still defined on the subsets of $M$. It is the generalization of a combinatorial auction (where $k = 1$).

Since the size of the input is exponential, we suppose that we have oracles for accessing it. There are two common types of query methods. One common type of queries is the “value queries”. Given a bundle of $S$, a value query answers $v(S)$ for a valuation $v$. From a “computer science” perspective, this kind of query is very natural.

Another kind of query is the “demand queries”. Given a vector $p = (p_1, \ldots, p_m)$ of item prices, a demand query replies a set that maximizes the profit, i.e. maximizes $v_i(S) - \Sigma_{j \in S} p_j$. Demand queries are very natural from an economic point of view. It is known that demand queries can simulate values queries in polynomial time [4].

In this paper we study the important cases where all bidders are known to have subadditive and submodular valuations; a valuation is called subadditive if $v(S \cup T) \leq v(S) + v(T)$ for all $S, T \subseteq M$; a valuation is called submodular if $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$ for all $S, T \subseteq M$. It is known that every submodular valuation is subadditive [17].

For general utility functions, the combinatorial auction problem is NP-hard. In [4], it has been shown that there are no polynomial time algorithms with a factor better than $O\left(\frac{\log m}{m}\right)$ if value queries are used. In [18] and [21], it has also been shown that there are no polynomial time algorithms with a factor better than $O\left(\frac{1}{m^{1/2-\epsilon}}\right)$ even for single minded bidders. If demand queries are used, achieving any approximation factors better than $O\left(\frac{1}{m^{1/2-\epsilon}}\right)$ requires exponential communication [20]. More results on the combinatorial auction problems with general utilities can be found in [7].

The allocation problem with subadditive utility functions is still NP-hard. Using demand queries, a $O\left(\log m\right)$ approximation algorithm for combinatorial auctions with subadditive utility function is given in [8]. In the same paper, an incentive compatible $O\left(\sqrt{m}\right)$ approximation algorithm is also presented if value queries is used. Recently, Feige give an approximation algorithm that obtains the approximation ratio of 2 ([12], [13]). As for complexity results, it is shown that an exponential amount of communication is required for achieving an approximation ratio better than 2 in [8]. In [12] and [13], it is proved that there are no polynomial time algorithms approximating the maximum welfare within a factor $2 - \epsilon$ unless $P = NP$, when bidders are subadditive. Thus the approximation ratio 2 is the best possible for the combinatorial auctions with subadditive utility functions.

In [17], a strict hierarchy of subclasses within the class of subadditive valuations is presented: $OXS \subset GS \subset SM \subset XOS \subset CF$. The $CF$ is the class of subadditive (complement-free) valuations; $SM$ is the set of submodular...
valuations; The XOS is the set of those valuations that can be defined by XOR-of-ORs of singleton valuations.

For the combinatorial auctions with XOS utility functions, a greedy algorithm achieving an approximation ratio of 2 is given in [8], [9]. An improved ratio of $\frac{\epsilon}{\epsilon-1}$ is obtained in [10] and [12]. It is shown that it is NP-hard to approximate the optimal allocation with XOS valuations to within any factor of $\frac{\epsilon}{\epsilon-1} - \epsilon$ [8], [9]. It is also proved that exponential communication is required for achieving any approximation ratio better than $\frac{\epsilon}{\epsilon-1}$ when all bidders are XOS.

In the past years, combinatorial auctions with submodular bidders have also received much attention. A greedy 2-approximation algorithm is given in [17] and the approximation ratio is improved to $(2 - \frac{1}{m})$ in [10] when value queries is used. In [22], Jan Vondrak design a randomized continuous greedy $\frac{\epsilon}{\epsilon-1}$-approximation algorithm for the submodular welfare problem in the value oracle model. In [1], Ittai Abraham and Moshe Babaioff et al. develop polynomial-time approximation algorithms and truthful mechanisms for welfare maximization with bidders with hypergraph valuations. When demand queries is used, there is a random polynomial time approximation algorithm that obtains an approximation ratio of $\rho < \frac{\epsilon}{\epsilon-1}$ in [14]. In [8], it is shown that it is NP-hard to approximate the optimal allocation for combinatorial auctions with submodular bidders to within a factor better than 51/50, unless $P = NP$. Khot et al. improve this result, prove that there are no polynomial time algorithms that can obtain an approximation ratio better than $\frac{\epsilon}{\epsilon-1}$ using value queries only, unless $P = NP$ [16]. The case of additive valuations with a budget limit is a subcase of submodular valuations. It is NP-hard to find the optimal allocation in a combinatorial auction with valuations that are additive with budget limit [17]. In [2], a randomized algorithm with an approximation ratio of $\frac{\epsilon}{\epsilon-1}$ is presented, which can be derandomized. Some other subcases of submodular valuations have also been studied (e.g. identical bidders [8], [9], online settings [19].)

In [5], an incentive compatible mechanism for multi-unit combinatorial auctions is given. In particular, this includes the case where each good has exactly $k$ units, i.e. $k$-duplicates combinatorial auctions. In [11], Dobzinski and Schapira exhibit a polynomial time $\min\{\frac{n}{\epsilon}, \mathcal{O}(m \frac{1}{\epsilon})\}$ approximation algorithm for $k$-duplicates combinatorial auctions using demand queries only and show that exponential communication is required for achieving an approximation ratio better than $\min\{\frac{n}{\epsilon}, \mathcal{O}(m \frac{1}{\epsilon})\}$, where $\epsilon > 0$. In the same paper, they also give an algorithm that achieves an approximation ratio of $\mathcal{O}(\sqrt{\log m})$ using only a polynomial number of value queries and prove that it is impossible to approximate a combinatorial auction with $k$-duplicates to a factor of $\mathcal{O}(\sqrt{\log m})$ using a polynomial number of value queries. They studied the case where all valuations are general utility functions. In [6], a $\mathcal{O}(\sqrt{m})$ approximation algorithm for $k$-duplicates combinatorial auctions with subadditive valuations using value queries and a $\mathcal{O}(\log m)$ approximation algorithm for $k$-duplicates combinatorial auctions with subadditive valuations using demand queries are given.

In this paper, we study the computational complexity of $k$-duplicates combinatorial auctions with subadditive bidders and approximation algorithms for $k$-duplicates combinatorial auctions with submodular bidders.

**Our Results**

In this paper, first, we prove some lower bounds for subadditive bidders. We show that it is NP-hard to approximate the maximum welfare for $k$-duplicates combinatorial auctions with subadditive bidders within a factor of $2 - \epsilon$ where $\epsilon > 0$ unless $P = NP$. Secondly, we provide some approximation algorithms. We give a $2$-approximation algorithm for $k$-duplicates combinatorial auctions with submodular bidders.

### Structure of the Paper

In section 2, we study the computational complexity for $k$-duplicates combinatorial auctions with subadditive valuations. In section 3 we present a $2$-approximation algorithm for $k$-duplicates combinatorial auctions in which all valuations are submodular. Finally, in section 4 we present some conclusions and some open problems.

### 2. The Complexity for $k$-duplicates Combinatorial Auctions with Subadditive Valuations

In this section we study the computational complexity for $k$-duplicates combinatorial auctions with subadditive valuations. In the following, we show that it is NP-hard to approximate the optimal allocation within a factor of $2 - \epsilon$ for $k$-duplicates combinatorial auctions with subadditive valuations, where $\epsilon > 0$.

In order to get the computational complexity, we will give a polynomial time reduction from the maximum independent set problem for hypergraphs to $k$-duplicates combinatorial auctions with subadditive valuations. First, we give two definitions as follows.

**Definition 1** A $k$-uniform hypergraph $H_k(V, E)$ consists of a set of vertices $V$ and a collection $E$ of $k$-element subsets of $V$ that are called hyperedges. An independent set of $H_k$ is a set of vertices such that no subset of these vertices form a hyperedge in $H_k$.

**Definition 2** The Maximum Independent Set (MIS) problem for Hypergraphs is the following problem: Given a hypergraph $H_k$, find a maximum independent set.
In [3], it is shown that for every $\epsilon > 0$, there is an $\alpha > 0$ such that it is NP-hard to distinguish between “yes cases” in which a graph has an independent set of size $\alpha n$ and “no cases” in which every independent set is of size at most $\epsilon \alpha n$. A gap-preserving reduction from the MIS problem for graphs to the MIS for $k$-uniform hypergraphs is given in [5] and [15]. Thus the following conclusion holds.

**Lemma 1** ([5] and [15]) Let $k \geq 2$ be a fixed integer, for every $\epsilon > 0$, there is an $\alpha > 0$ such that it is NP-hard to distinguish between “yes cases” in which a $k$-uniform hypergraph has an independent set of size $\alpha n$ and “no cases” in which every independent is of size at most $\epsilon \alpha n$.

In the following, we give a reduction from the MIS problem for $k+1$-hypergraph to the $k$-duplicates combinatorial auctions with subadditive valuations. The reduction is the extension of Feige’ reduction and is similar to that in [5], where it is shown that it is NP-hard to approximate $k$-duplicates combinatorial auctions to within a factor of $O(m \frac{1}{1+\epsilon})$ unless $NP = ZPP$, for every fixed $k \geq 1$ and $\epsilon > 0$.

**Theorem 1** For every $\epsilon > 0$, it is NP-hard to approximate the optimal allocation within a factor of $2 - \epsilon$ for $k$-duplicates combinatorial auctions with subadditive valuations.

**Proof:** By lemma 1, it is NP-hard to distinguish between “yes cases” in which a $k+1$-uniform hypergraph has an independent set of size $\alpha n$ and “no cases” in which every independent set is of size at most $\epsilon \alpha n$. Given an instance of $k+1$-uniform hypergraph $H_{k+1} = (V, E)$, we define an instance of $k$-duplicates combinatorial auctions with subadditive valuations as follows: Let the hyperedges of the hypergraph $H_{k+1}$ be the items with $k$ duplicates of each item and let the number of bidders be $\alpha n$. Every bidder has the same subadditive valuation. The subadditive valuation is defined as follows. Suppose $S$ is a subset of items. If there is some vertex such that $S$ contains all items whose corresponding hyperedges are incident with it, the utility $v(S) = 2$. Otherwise, $v(S) = 1$. We show that the utility function is subadditive. Let $S$ and $T$ be any two subset of items. By definition of $v$, $v(S) + v(T) \geq 1 + 1 = 2$ and $v(S \cup T) \leq 2$. So $v(S \cup T) \leq v(S) + v(T)$. Thus the utility function $v$ is a subadditive valuation. On yes cases, by giving each bidder the items corresponding to hyperedges incident with some vertex of a maximum independent set, the maximum welfare is $2\alpha n$. On no cases, the maximum welfare is at most $(1 + \epsilon)\alpha n$. The reason is as follows. Since the maximum independent set is of size at most $\epsilon \alpha n$ on no cases, the number of its corresponding bidders is $\alpha n$. We assign each such bidder the items corresponding to hyperedges incident with that vertex. Thus there are $2\alpha n$ welfare for these $\alpha n$ bidders; For other $(1 - \epsilon)\alpha n$ bidders, if one such bidder is assigned to all items whose corresponding hyperedges are incident with it, then the vertex corresponding to that bidder and the maximum independent set contain a hyperedge. Thus the item corresponding to the hyperedge is assigned to $k+1$ different bidders, this contradict the fact that the duplicate of each item is $k$. Thus for other $(1 - \epsilon)\alpha n$ bidders, there are no bidder that are assigned to all items whose corresponding hyperedges are incident with it. Thus their welfare are at most $(1 - \epsilon)\alpha n$. So on no cases, the welfare are at most $2\alpha n + (1 - \epsilon)\alpha n = (1 + \epsilon)\alpha n$. Thus the hardness factor is $\frac{2\alpha n}{(1+\epsilon)\alpha n} = 2 - \epsilon$.

3. A 2-approximation algorithm for submodular valuations

In this section we present a 2-approximation algorithm for $k$-duplicates combinatorial auctions in which all valuations are submodular, which extends the algorithm of [17]. In [17], an equivalent definition of submodular valuations is as follows: for all $S \subseteq T$ and $x \notin T$, $v(S \cup x) - v(S) \geq v(T \cup x) - v(T)$. That is, the marginal value of each item decreases as the set of items already acquired increases. The approximation algorithm is as follows.

**Input:** $v_1, \ldots, v_n$-submodular valuations, given as black boxes.

**Output:** An allocation $S_1, \ldots, S_n$ which is 2-approximation to the optimal allocation.

**The Algorithm:**

1. Initialize $S_1 = S_2 = \ldots = S_n = \emptyset$.
2. For $x = 1, \ldots, m$ do:
   a) Let $j_1, \ldots, j_k$ be the bidders whose value of $v_j(x|S_j)$ is the highest, the second high, \ldots, the $k$-th high, where $v_j(x|S_j) = v_j(S_j \cup x) - v_j(S_j)$.
   b) Allocate $x$ to $j_1, \ldots, j_k$, i.e. $S_{j_1} \leftarrow S_{j_1} \cup \{x\}, \ldots, S_{j_k} \leftarrow S_{j_k} \cup \{x\}$.

   Obviously, the above algorithm requires only a polynomial number of operations.

**Theorem 3** The above algorithm provides a 2-approximation to the optimal allocation for $k$-duplicates combinatorial auctions in which all valuations are submodular.

**Proof:** We denote by $Q$ the original problem and define a new problem $Q'$ on the $m-1$ remaining items with $k$-duplicates of each item after item 1 is removed; i.e., item 1 is unavailable and $v_{j_1}, \ldots, v_{j_k}$ are replaced by $v'_{j_1}, \ldots, v'_{j_k}$ with $v'_{j_1}(S) = v_{j_1}(S \setminus \{1\}) = v_{j_1}(S \cup \{1\}) - v_{j_1}(\{1\}), \ldots, v'_{j_k}(S) = v_{j_k}(S \setminus \{1\}) = v_{j_k}(S \cup \{1\}) - v_{j_k}(\{1\})$, where $j_1, \ldots, j_k$ are those bidders to whom item
1 was allocated. All other valuations \( v_i, i \neq j_1, \ldots, j_k \) are unchanged. Notice that the above algorithm can be viewed as first item 1 is allocated to \( j_1, \ldots, j_k \) and other items are assigned using a recursive call on \( Q' \).

Let \( ALG(Q) \) denote the value of the allocation produced by above algorithm and \( OPT(Q) \) denote the value of optimal allocation. Let \( p_1 = v_{j_1}(\{1\}), \ldots, p_k = v_{j_k}(\{1\}) \). By the definition of \( Q' \), we have \( v_{j_1}'(S) + v_{j_1}(\{1\}) = v_{j_1}(S \cup \{1\}), \ldots, v_{j_k}'(S) + v_{j_k}(\{1\}) = v_{j_k}(S \cup \{1\}) \). So we get \( ALG(Q) = ALG(Q') + p_1 + \cdots + p_k \) We will show that \( OPT(Q) \leq OPT(Q') + 2(p_1 + \cdots + p_k) \). Let \( S_1, \ldots, S_n \) be the optimal allocation \( S \) for \( Q \), and assume that \( 1 \in S_{i_1}, \ldots, 1 \in S_{i_k} \) and \( v_{j_k}(\{1\}) \geq v_{j_1}(\{1\}) \geq \cdots \geq v_{j_k}(\{1\}) \). Let \( S' \) be the allocation of item 2, \ldots, \( m \) that is the remaining allocation when \( k \)-duplicates of item 1 are removed in the allocation \( S \). This is a possible solution to \( Q' \). Let us compare \( OPT(Q') \) to \( OPT(Q) \). All bidders except \( j_1, \ldots, j_k \) get the same allocation and all bidders except \( j_1, \ldots, j_k \) have the same valuation. Without loss of generality, assume that \( l_1 \neq j_1, \ldots, l_k \neq j_k \). Since \( v_{l_i}(S \cup \{1\}) - v_{l_i}(S) \leq v_{l_i}(\{1\})(i = 1, \ldots, k) \), the bidders \( l_1, \ldots, l_k \) lose at most \( v_{l_i}(\{1\}) = p_i \) for all \( i = 1, \ldots, k \). Thus the bidders \( l_1, \ldots, l_k \) lose at most \( p_1 + \cdots + p_k \). By the monotonicity of \( v_{j_i}(i = 1, \ldots, k) \), \( v_{j_i}'(S_{j_i}) \geq v_{j_i}(S_{j_i} \cup \{1\}) - v_{j_i}(\{1\}) \). Therefore \( OPT(Q') \leq OPT(Q) - 2(p_1 + \cdots + p_k) \).

By lemma 1 of [17], \( Q' \) also consists of submodular valuations. Thus the proof is concluded by induction on \( Q' \): \( OPT(Q) \leq OPT(Q') + 2(p_1 + \cdots + p_k) \leq 2ALG(Q') + 2(p_1 + \cdots + p_k) \leq 2ALG(Q) \).

4. Conclusions

In this paper, we have studied the computational complexity for \( k \)-duplicates combinatorial auctions with subadditive valuations. What are the computational complexity for \( k \)-duplicates combinatorial auctions with submodular or \( XOS \) valuations? The problem should be further studied. About approximation algorithms, how to improve the approximation ratio 2 for submodular valuations. What is the upper bound for \( k \)-duplicates combinatorial auctions with subadditive or \( XOS \) valuations?

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