Equivalence of the Foulis-Holland Theorems and the Orthomodular Law in Quantum Logic: Part 3

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Abstract

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of quantum-mechanical behaviors and operations. In much the same way that the structure of conventional propositional (Boolean) logic (BL) is the logic of the description of the behavior of classical physical systems and is isomorphic to a Boolean algebra (BA), so also the algebra, $C(H)$, of closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space is a logic of the descriptions of the behavior of quantum mechanical systems and is a model of an ortholattice (OL). An OL can thus be thought of as a kind of “quantum logic” (QL). $C(H)$ is also a model of an orthomodular lattice (OML), which is an ortholattice to which the orthomodular law has been conjoined. Now a QL can be thought of as a BL in which the distributive law does not hold. Under certain commutativity conditions, a QL does satisfy the distributive law; among the most well known of these relationships are the Foulis-Holland theorems (FHTs). Here I provide an automated deduction of one of the four FHTs from OML.

Keywords: automated deduction, quantum computing, orthomodular lattice, Foulis-Holland theorems, Hilbert space

1.0 Introduction

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of the description of quantum-mechanical behaviors ([1], [17], [18], [20]). In much the same way that conventional propositional (Boolean) logic (BL,[12]) is the logical structure of description of the behavior of classical physical systems (e.g. “the measurements of the position and momentum of particle P are commutative”, i.e., can be measured in either order, yielding the same results) and is isomorphic to a Boolean lattice ([10], [11], [19]), so also the algebra, $C(H)$, of the closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space $H$ ([1], [4], [6], [9], [13]) is a logic of the descriptions of the behavior of quantum mechanical systems (e.g., “the measurements of the position and momentum of particle P are not commutative”) and is a model ([10]) of an ortholattice (OL; [4]). An OL can thus be thought of as a kind of “quantum logic” (QL; [19]). Figure 1 shows a set of axioms for an ortholattice.
Lattice axioms

\[ x = c(c(x)) \quad \text{(AxLat1)} \]
\[ x \lor y = y \lor x \quad \text{(AxLat2)} \]
\[ (x \lor y) \lor z = x \lor (y \lor z) \quad \text{(AxLat3)} \]
\[ (x \land y) \land z = x \land (y \land z) \quad \text{(AxLat4)} \]
\[ x \lor (x \land y) = x \quad \text{(AxLat5)} \]
\[ x \land (x \lor y) = x \quad \text{(AxLat6)} \]

Ortholattice axioms

\[ c(x) \land x = 0 \quad \text{(AxOL1)} \]
\[ c(x) \lor x = 1 \quad \text{(AxOL2)} \]
\[ x \land y = c(c(x) \lor c(y)) \quad \text{(AxOL3)} \]

A useful definition

\[ 1_2 = y \lor ((x \land c(y)) \lor (c(x) \land c(y))) \]

where

- \( x, y \) are variables ranging over lattice nodes
- \( ^\land \) is lattice meet
- \( \lor \) is lattice join
- \( c(x) \) is the orthocomplement of \( x \)
- \(<->\) means if and only if
- \( = \) is equivalence \((\text{[12]})\)
- \( 1 \) is the maximum lattice element \((- x \lor c(x))\)
- \( 0 \) is the minimum lattice element \((- c(1))\)

Figure 1. Lattice, ortholattice, ortholattice axioms, and a useful definition.

\( C(H) \) is also a model of an orthomodular lattice \((\text{OML}; [4], [7])\), which is an OL conjoined with the orthomodularity axiom \((\text{OMLaw})\):

\[ y \lor (c(y) \land (x \lor y)) = x \lor y \quad \text{OMLaw} \]

The rationalization of the OMA as a claim proper to physics has proven problematic \(([13], \text{Section 5-6})\), motivating the question of whether the OMA is required in an adequate characterization of QL. Thus formulated, the question suggests that the OMA and its equivalents are specific to an OML, and that as a consequence, banning the OMA from QL yields a "truer" quantum logic.

Now a QL can be thought of as a BL in which the distributive law

\[ (\text{D}) \quad (x \lor (y \land z)) = (x \lor y) \land (x \lor z) \]

does not hold. Under certain commutativity conditions, a QL does satisfy the distributive law; among the most well-known of these relationships are the Foulis-Holland theorems \((\text{FHTs ([7])})\):
Foulis-Holland theorem FH1
(C(x,y) & C(x,z)) -> ( (x ^ (y v z)) = ((x ^ y) v (x ^ z)) )

Foulis-Holland theorem FH2
(C(x,y) & C(x,z)) -> ( (y ^ (x v z)) = ((y ^ x) v (y ^ z)) )

Foulis-Holland theorem FH3
(C(x,y) & C(x,z)) -> ( (x v (y ^ z)) = ((x v y) ^ (x v z)) )

Foulis-Holland theorem FH4
(C(x,y) & C(x,z)) -> ( (y v (x ^ z)) = ((y v x) ^ (y v z)) )

where C(x,y), "x commutes with y" is defined as
C(x,y) <-> (x = ((x ^ y) v (x ^ c(y))))

Figure 2. The Foulis-Holland theorems.

2.0 Method

The OML axiomatizations of Megill, Pavičić, and Horner ([5], [14], [15], [16], [21]) were implemented in a prover9 ([2]) script ([3]) configured to derive FH3, then executed in that framework on a Dell Inspiron 545 with an Intel Core2 Quad CPU Q8200 (clocked @ 2.33 GHz) and 8.00 GB RAM, running under the Windows Vista Home Premium/Cygwin operating environment.

3.0 Results

Figure 3 shows the proof, generated by [3] on the platform described in Section 2.0, that FH3 is implied by an OML:
76 x v c(x v c(y v x)) = y v x.  [copy(75), rewrite([23(3),14(2)])].
77 c(c(c1 v c2) v c(c1 v c3)) != (c1 v c2) v c(c3)) # answer("Foulis-Holland Theorem 3").  [deny(4)].
78 c(c(c1 v c2) v c(c(c1 v c2) v c(c3))) != (c1 v c2) v (c(c1 v c3)) # label("Foulis-Holland Theorem 3") # answer("Foulis-Holland Theorem 3").  [copy(77), rewrite([23(3),14(2)]), flip(a)].
83 c(1) = 0.  [back_rewrite(20), rewrite([23(2),14(2),22(2)])].
84 c(c(x) v c(x v y)) = x.  [back_rewrite(19), rewrite([23(2)])].
85 x v c(c(x) v c(y)) = x.  [back_rewrite(18), rewrite([23(1)])].
89 x v (y v z) = y v (x v z).  [para(15(a,1),16(a,1)), rewrite([16(2)])].
98 x v (y v c(x v y)) = 1.  [para(22(a,1),16(a,1)), flip(a)].
99 x v (c(x v y) v c(c(y) v x)) = 1/2.  [para(15(a,1),68(a,1,2,1,1,1))].
108 x v c(x v c(x v y)) = y v x.  [para(15(a,1),76(a,1,2,1,2,1))].
110 x v (y v c(x v (y v c(z v (x v y))))) = z v (x v y).  [para(76(a,1),16(a,1)), rewrite([16(7)])].
113 1/2 = 1.  [para(76(a,1),68(a,1,2,1,1,1)), rewrite([84(13),15(7),15(8),22(8)])].
124 x v (c(x v y) v c(c(y) v z)) = x v z.  [para(146(a,1),16(a,1,1)), flip(a)].
133 c(x) v c(y v x) = c(x).  [para(84(a,1),14(a,1,1)), flip(a)].
137 c(0 v c(x)) = x.  [para(22(a,1),84(a,1,2,1,2,1)), rewrite([83(3),15(3)])].
141 1 v x = 1.  [para(83(a,1),84(a,1,1,2,1)), rewrite([137(6)])].
146 x v c(c(x) v y) = x.  [para(14(a,1),85(a,1,2,1,2,1))].
150 x v 0 = x.  [para(22(a,1),85(a,1,2,1,1)), rewrite([83(2)])].
151 x v c(c(x) v c(x v z)) = x.  [para(76(a,1),85(a,1,2,1))].
165 x v (y v c(x v c(x v y))) = y v (x v z).  [para(76(a,1),89(a,1,2,1,2))].
196 0 v x = x.  [para(150(a,1),15(a,1)), flip(a)].
215 x v c(c(x) v y) v z = x v z.  [para(146(a,1),16(a,1,1)), flip(a)].
220 c(x) v c(y v x) = c(x).  [para(14(a,1),15(a,1,2,1,2,1))].
230 x v (y v c(x v y)) = x.  [para(15(a,1),98(a,1,2,2,1,1))].
234 c(x) v (y v c(x v z)) = y v c(x).  [para(133(a,1),89(a,1,2), flip(a)].
259 c(x v y) v (c(c(x) v y) v z) = c(y).  [para(220(a,1),16(a,1,1)), rewrite([141(2),16(5)])].
260 c(x) v (y v c(z v x)) = y v c(x).  [para(220(a,1),89(a,1,2), flip(a)].
262 x v (y v (c(y v x)) v z) = 1.  [para(230(a,1),16(a,1,1)), flip(a)].
326 c(c(x) v y) v (z v x) = z v x.  [para(146(a,1),110(a,1,2,1,2,1,2,1,2,1)), rewrite([215(10),165(9),146(9)])].
741 x v c(c(x v c(x v y)) v z) = 1.  [para(98(a,1),124(a,1,2,1,1)), rewrite([83(2),15(6),196(8)])].
855 c(x) v (y v c(c(x v u) v y)) = c(x) v (y v u).  [para(326(a,1),215(a,1,2,1,1,1)), rewrite([16(6),16(9)])].
3700 x v c(y v c(x v y)) = x.  [para(741(a,1),108(a,1,2,1,1)), rewrite([83(2),150(2,15(5))], flip(a)].
3710 x v c(y v c(y v x)) = x.  [para(15(a,1),3700(a,1,2,1,2,1))].
3719 x v c(y v x) = x v c(y).  [para(3700(a,1),15(a,1,2,1,1)), rewrite([15(5),236(5)]), flip(a)].
3992 x v c(x v y) v c(y v c(x v y)) = c(y).  [back_rewrite(259), rewrite([3719(5)])].
4092 x v c(x v y) = x v c(y).  [para(3710(a,1),15(a,1,2,1,1)), rewrite([15(5),260(5)]), flip(a)].
4098 x v (c(x v y) v z) = c(y) v (x v z).  [para(262(a,1),3710(a,1,2,1,2,1,1)), rewrite([83(6),150(6),15(6),855(6)]), flip(a)].
4422 c(c(x v y) v z) = c(c(x) v z) v c(c(y) v (x v z)).  [para(234(a,1),3992(a,1,1,1)), rewrite([141(8),15(7),4098(7)])].
4544 $F \# answer("Foulis-Holland Theorem 3")$.  [back_rewrite(78), rewrite([4422(10),133(7),14(3),4092(9),89(8),4092(10)]), xx(a)].

Figure 3. Summary of a prover9 ([2]) proof of FH3 from OML. The proofs assume the default inference rules of prover9. The general form of a line in this proof is “line_number conclusion [derivation]”, where line_number is a unique identifier of a line in the proof, and conclusion is the result of applying the prover9 inference rules (such as paramodulation, copying, and rewriting), noted in square brackets (denoting the derivation), to the lines cited in those brackets. Note that some of “logical” proof lines in the above have been transformed to two text lines, with the derivation appearing on a text line following a text line containing the first part of that logical line. The detailed syntax and semantics of these notations can be found in [2]. All prover9 proofs are by default proofs by contradiction.
The total time to produce the proofs in Figure 3 on the platform described in Section 2.0 was approximately 2.4 seconds.

4.0 Discussion

The results of Section 3.0 motivate several observations:

1. FH3 is derivable from OML.

2. The proof in Section 3.0 is, as far as I know, novel.

3. Companion papers provide derivations of the remaining FHTs from OML, and a derivation of the OMLaw from an OML without the FHTs, conjoined with the FHTs. The union of these proofs constitutes a proof of the equivalence of the OMLaw and the FHTs within OML theory.

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6.0 References


