Equivalence of the Foulis-Holland Theorems and the Orthomodular Law in Quantum Logic: Part 3

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Abstract

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of quantum-mechanical behaviors and operations. In much the same way that the structure of conventional propositional (Boolean) logic (BL) is the logic of the description of the behavior of classical physical systems and is isomorphic to a Boolean algebra (BA), so also the algebra, C(H), of closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space is a logic of the descriptions of the behavior of quantum mechanical systems and is a model of an ortholattice (OL). An OL can thus be thought of as a kind of "quantum logic" (QL). C(H) is also a model of an orthomodular lattice (OML), which is an ortholattice to which the orthomodular law has been conjoined. Now a QL can be thought of as a BL in which the distributive law does not hold. Under certain commutativity conditions, a QL does satisfy the distributive law; among the most well known of these relationships are the Foulis-Holland theorems (FHTs). Here I provide an automated deduction of one of the four FHTs from OML.

Keywords: automated deduction, quantum computing, orthomodular lattice, Foulis-Holland theorems, Hilbert space

1.0 Introduction

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of the description of quantum-mechanical behaviors ([1], [17], [18], [20]). In much the same way that conventional propositional (Boolean) logic (BL,[12]) is the logical structure of description of the behavior of classical physical systems (e.g. "the measurements of the position and momentum of particle P are commutative", i.e., can be measured in either order, yielding the same results) and is
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isomorphic to a Boolean lattice ([10], [11], [19]), so also the algebra, *C(H)*, of the closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space *H* ([1], [4], [6], [9], [13]) is *a* logic of the descriptions of the behavior of quantum mechanical systems (e.g., "the measurements of the position and momentum of particle P are *not* commutative") and is a model ([10]) of an ortholattice (OL; [4]). An OL can thus be thought of as a kind of "quantum logic" (QL; [19]). Figure 1 shows a set of axioms for an orthlolattice

Lattice axioms

```
x = c(c(x))                    (AxLat1)
x v y = y v x                  (AxLat2)
(x v y) v z = x v (y v z)      (AxLat3)
(x ^ y) ^ z = x ^ (y ^ z)      (AxLat4)
x v (x ^ y) = x                (AxLat5)
x ^ (x v y) = x                (AxLat6)
```

Ortholattice axioms

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c(x) ^ x = 0                   (AxOL1)
c(x) v x = 1                   (AxOL2)
x ^ y = c(c(x) v c(y))         (AxOL3)
```

A useful definition

```
1_2 = y v ((x ^ c(y)) v (c(x) ^ c(y)))
```

```
where
   x, y are variables ranging over lattice nodes
   ^ is lattice meet
   v is lattice join
   c(x) is the orthocomplement of x
   <-> means if and only if
   =  is equivalence ([12])
   1 is the maximum lattice element (= x v c(x))
   0 is the minimum lattice element (= c(1))
```

Figure 1. Lattice, ortholattice, ortholattice axioms, and a useful definition.

C(H) is also a model of an orthomodular lattice (OML; [4], [7]), which is an OL conjoined with the orthomodularity axiom (OMLaw):

```
y v (c(y) ^ (x v y)) =
     x v y        (OMLaw)
```

The rationalization of the OMA as a claim proper to physics has proven problematic ([13], Section 5-6), motivating the question of whether the OMA is required in an adequate characterization of QL. Thus formulated, the question suggests that the OMA and its equivalents are specific to an OML, and that as a consequence, banning the OMA from QL yields a "truer" quantum logic.

Now a QL can be thought of as a BL in which the distributive law

```
(D)    (x v (y ^ z) =
          (x v y) ^ (x v z))
```

does not hold. Under certain commutativity conditions, a QL does satisfy the distributive law; among the most well known of these relationships are the Foulis-Holland theorems (FHTs ([7])):

```
% Foulis-Holland theorem FH1
(C(x,y) & C(x,z)) ->  ( (x ^ (y v z)) = ((x ^ y) v (x ^ z))  )

% Foulis-Holland theorem FH2
(C(x,y) & C(x,z)) ->  ( (y ^ (x v z)) = ((y ^ x) v (y ^ z))  )


% Foulis-Holland theorem FH3
(C(x,y) & C(x,z)) ->  ( (x v (y ^ z)) = ((x v y) ^ (x v z))  )

% Foulis-Holland theorem FH4
(C(x,y) & C(x,z)) ->  ( (y v (x ^ z)) = ((y v x) ^ (y v z))  )


where C(x,y), "x commutes with y" is defined as

        C(x,y) <-> (x = ((x ^ y) v (x ^ c(y))))
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Figure 2. The Foulis-Holland theorems.

2.0 Method

The OML axiomatizations of Megill, Pavičić, and Horner ([5], [14], [15], [16], [21]) were implemented in a *prover9* ([2]) script ([3]) configured to derive FH3, then executed in that framework on a Dell Inspiron 545 with an Intel Core2 Quad CPU Q8200 (clocked @ 2.33 GHz) and 8.00 GB RAM, running under the *Windows Vista Home Premium /Cygwin* operating environment.

3.0 Results

Figure 3 shows the proof, generated by [3] on the platform described in Section 2.0, that FH3 is implied by an OML:

```
============================== PROOF =================================

% Proof 1 at 2.32 (+ 0.11) seconds: "Foulis-Holland Theorem 3".
% Length of proof is 54.

4 C(x,y) & C(x,z) -> x v (y ^ z) = (x v y) ^ (x v z) # label("Foulis-Holland Theorem 3")
# label(non_clause) # label(goal).  [goal].
13 x = c(c(x)) # label("AxL1").  [assumption].
14 c(c(x)) = x.  [copy(13),flip(a)].
15 x v y = y v x # label("AxL2").  [assumption].
16 (x v y) v z = x v (y v z) # label("AxL3").  [assumption].
18 x v (x ^ y) = x # label("AxL5").  [assumption].
19 x ^ (x v y) = x # label("AxL6").  [assumption].
20 c(x) ^ x = 0 # label("AxOL1").  [assumption].
21 c(x) v x = 1 # label("AxOL2").  [assumption].
22 x v c(x) = 1.  [copy(21),rewrite([15(2)])].
23 x ^ y = c(c(x) v c(y)) # label("AxOL3").  [assumption].
67 1_2 = x v ((y ^ c(x)) v (c(y) ^ c(x))) # label("Df. 2.20").  [assumption].
68 x v (c(y v x) v c(c(y) v x)) = 1_2.
[copy(67),rewrite([23(3),14(4),23(7),14(6),14(6),15(7)]),flip(a)].
75 x v (c(x) ^ (y v x)) = y v x # label("OMLaw").  [assumption].
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```
76 x v c(x v c(y v x)) = y v x.  [copy(75),rewrite([23(3),14(2)])].
77 c1 v (c2 ^ c3) != (c1 v c2) ^ (c1 v c3) # label("Foulis-Holland Theorem 3") #
answer("Foulis-Holland Theorem 3").  [deny(4)].
78 c(c(c1 v c2) v c(c1 v c3)) != c1 v c(c(c2) v c(c3)) # answer("Foulis-Holland Theorem
3").  [copy(77),rewrite([23(4),23(15)]),flip(a)].
83 c(1) = 0.  [back_rewrite(20),rewrite([23(2),14(2),22(2)])].
84 c(c(x) v c(x v y)) = x.  [back_rewrite(19),rewrite([23(2)])].
85 x v c(c(x) v c(y)) = x.  [back_rewrite(18),rewrite([23(1)])].
89 x v (y v z) = y v (x v z).  [para(15(a,1),16(a,1,1)),rewrite([16(2)])].
98 x v (y v c(x v y)) = 1.  [para(22(a,1),16(a,1)),flip(a)].
99 x v (c(x v y) v c(c(y) v x)) = 1_2.  [para(15(a,1),68(a,1,2,1,1))].
108 x v c(x v c(x v y)) = y v x.  [para(15(a,1),76(a,1,2,1,2,1))].
110 x v (y v c(x v (y v c(z v (x v y))))) = z v (x v y).
[para(76(a,1),16(a,1)),rewrite([16(7)]),flip(a)].
113 1_2 = 1.  [para(76(a,1),68(a,1,2,1,1)),rewrite([84(13),15(7),15(8),22(8)]),flip(a)].
124 x v (c(x v y) v c(c(y) v x)) = 1.  [back_rewrite(99),rewrite([113(8)])].
133 c(x) v c(x v y) = c(x).  [para(84(a,1),14(a,1,1)),flip(a)].
137 c(0 v c(x)) = x.  [para(22(a,1),84(a,1,1,2,1)),rewrite([83(3),15(3)])].
141 1 v x = 1.  [para(83(a,1),84(a,1,1,1)),rewrite([137(6)])].
146 x v c(c(x) v y) = x.  [para(14(a,1),85(a,1,2,1,2))].
150 x v 0 = x.  [para(22(a,1),85(a,1,2,1)),rewrite([83(2)])].
151 x v c(y v c(x)) = x.  [para(76(a,1),85(a,1,2,1))].
165 x v (y v c(x v c(z v x))) = y v (z v x).  [para(76(a,1),89(a,1,2)),flip(a)].
196 0 v x = x.  [para(150(a,1),15(a,1)),flip(a)].
215 x v (c(c(x) v y) v z) = x v z.  [para(146(a,1),16(a,1,1)),flip(a)].
220 c(x) v c(y v x) = c(x).  [para(14(a,1),151(a,1,2,1,2))].
230 x v (y v c(y v x)) = 1.  [para(15(a,1),98(a,1,2,2,1))].
234 c(x) v (c(x v y) v z) = c(x) v z.  [para(133(a,1),16(a,1,1)),flip(a)].
236 c(x) v (y v c(x v z)) = y v c(x).  [para(133(a,1),89(a,1,2)),flip(a)].
259 c(x v y) v c(y v c(x v y)) = c(y).
[para(220(a,1),76(a,1,2,1,2,1)),rewrite([14(6),15(5),220(11)])].
260 c(x) v (y v c(z v x)) = y v c(x).  [para(220(a,1),89(a,1,2)),flip(a)].
262 x v (y v (c(y v x) v z)) = 1.
[para(230(a,1),16(a,1,1)),rewrite([141(2),16(5)]),flip(a)].
326 c(c(x) v y) v (z v x) = z v x.
[para(146(a,1),110(a,1,2,2,1,2,2,1,2)),rewrite([215(10),165(9),146(9)])].
741 x v c(x v c(y v c(x v y))) = 1.
[para(98(a,1),124(a,1,2,1,1)),rewrite([83(2),15(6),196(8)])].
855 c(x) v (y v (c(z v x) v u)) = c(x) v (y v u).
[para(326(a,1),215(a,1,2,1,1)),rewrite([16(6),16(9)])].
3700 x v c(y v c(x v y)) = x.
[para(741(a,1),108(a,1,2,1)),rewrite([83(2),150(2),15(5)]),flip(a)].
3710 x v c(y v c(y v x)) = x.  [para(15(a,1),3700(a,1,2,1,2,1))].
3719 x v c(y v x) = x v c(y).
[para(3700(a,1),151(a,1,2,1)),rewrite([15(5),236(5)]),flip(a)].
3992 c(x v y) v c(y v c(x)) = c(y).  [back_rewrite(259),rewrite([3719(5)])].
4092 x v c(x v y) = x v c(y).
[para(3710(a,1),151(a,1,2,1)),rewrite([15(5),260(5)]),flip(a)].
4098 x v (c(x v y) v z) = c(y) v (x v z).
[para(262(a,1),3710(a,1,2,1,2,1)),rewrite([83(6),150(6),15(6),855(6)]),flip(a)].
4422 c(c(x v y) v z) = c(c(x) v z) v c(c(y) v (x v z)).
[para(234(a,1),3992(a,1,1,1)),rewrite([14(8),15(7),4098(7)]),flip(a)].
4544 $F # answer("Foulis-Holland Theorem 3").
[back_rewrite(78),rewrite([4422(10),133(7),14(3),4092(9),89(8),4092(10)]),xx(a)].

============================== end of proof ==========================
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Figure 3. Summary of a *prover9* ([2]) proof of FH3 from OML. The proofs assume the default inference rules of *prover9*. The general form of a line in this proof is "*line_number conclusion [derivation]*", where *line_number* is a unique identifier of a line in the proof, and *conclusion* is the result of applying the *prover9* inference rules (such as *para*modulation, *copy*ing, and *rewrit*ing), noted in square brackets (denoting the *derivation*), to the lines cited in those brackets. Note that some of "logical" proof lines in the above have been transformed to two text lines, with the *derivation* appearing on a text line following a text line containing the first part of that logical line. The detailed syntax and semantics of these notations can be found in [2]. All *prover9* proofs are by default proofs by contradiction.

The total time to produce the proofs in Figure 3 on the platform described in Section 2.0 was approximately 2.4 seconds.

4.0 Discussion

The results of Section 3.0 motivate several observations:

1. FH3 is derivable from OML.

2. The proof in Section 3.0 is, as far as I know, novel.

3. Companion papers provide derivations of the remaining FHTs from OML, and a derivation of the OMLaw from an OML without the OMLaw, conjoined with the FHTs. The union of these proofs constitutes a proof of the equivalence of the OMLaw and the FHTs within OML theory.

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6.0 References

[1] von Neumann J. *Mathematical Foundations of Quantum Mechanics*. 1936. Translated by R. T. Beyer. Princeton. 1983.

[2] McCune WW. *prover9 and mace4.* URL http://www.cs.unm.edu/~mccune/prover9/. 2009.

[3] Horner JK. *prover9* scripts for FH3. 2011. Available from the author on request.

[4] Dalla Chiara ML and Giuntini R. *Quantum Logics*. URL http://xxx.lanl.gov/abs/quant-ph/0101028. 2004.

[5] Megill ND and Pavičić M. Orthomodular lattices and quantum algebra. *International Journal of Theoretical Physics* 40 (2001), pp. 1387-1410.

[6] Akhiezer NI and Glazman IM. *Theory of Linear Operators in Hilbert Space. Volume I*. Translated by M. Nestell. Frederick Ungar. 1961.

[7] Holland, Jr. SS Orthomodularity in infinite dimensions: a theorem of M. Solèr. *Bulletin of the American Mathematical Society* 32 (1995), pp. 205-234.

[8] Marsden EL and Herman LM. *A condition for distribution in orthomodular lattices.* Kansas State University Technical Report #40. 1974.

[9] Knuth DE and Bendix PB. Simple word problems in universal algebras. In J. Leech, ed. *Computational Problems in Abstract Algebra*. Pergamon Press. 1970. pp. 263-297.

[10] Chang CC and Keisler HJ. *Model Theory*. North-Holland. 1990. pp. 38-39.

[11] Birkhoff G. *Lattice Theory*. Third Edition. American Mathematical Society. 1967.

[12] Church A. *Introduction to Mathematical Logic. Volume I.* Princeton. 1956.

[13] Jauch J. *Foundations of Quantum Mechanics.* Addison-Wesley. 1968.

[14] Megill ND. *Metamath.* URL http://us.metamath.org/qlegif/mmql.html#unify. 2004.

[15] Horner JK. An automated deduction system for orthomodular lattice theory.

Proceedings of the 2005 International Conference on Artificial Intelligence. CSREA Press. 2005. pp. 260-265.

[16] Horner JK. An automated equational logic deduction of join elimination in orthomodular lattice theory. *Proceedings of the 2007 International Conference on Artificial Intelligence.* CSREA Press. 2007. pp. 481-488.

[17] Messiah A. *Quantum Mechanics*. Dover. 1958.

[18] Horner JK. Using automated theorem-provers to aid the design of efficient compilers for quantum computing. *Los Alamos National Laboratory Quantum Institute Workshop.* December 9–10, 2002. URL http://www.lanl.gov/science/centers/quantum/qls_pdfs/horner.pdf.

[19] Birkhoff G and von Neumann J. The logic of quantum mechanics. *Annals of Mathematics* 37 (1936), 823-243.

[20] Nielsen MA and Chuang L . *Quantum Computation and Quantum Information*. Cambridge. 2000.

[21] Pavičić M and Megill N. Quantum and classical implicational algebras with primitive implication. *International Journal of Theoretical Physics* 37 (1998), 2091-2098. ftp://m3k.grad.hr/pavicic/quantum-logic/1998-int-j-theor-phys-2.ps.gz.