Intuitionistic Fuzzy Bi-Ideal of a Ring

P.K. Sharma¹, Aradhna Duggal²
¹P.G. Department of Mathematics, D.A.V. College, Jalandhar City, Punjab, India
²Department of Mathematics, S.G.G.S. Khalsa College, Mahil Pur, Punjab, India

Abstract - In this paper, the notion of intuitionistic fuzzy bi-ideal of a ring are defined and discussed. Some of their properties are studied.

Keywords: Intuitionistic fuzzy set (IFS), Intuitionistic fuzzy subring (IFSR), Intuitionistic fuzzy ideal (IFI), Intuitionistic fuzzy bi-ideal (IFBI)

1 Introduction


The notion of intuitionistic fuzzy set (IFS) was introduced by Atanassov [1] as a generalization of Zadeh’s fuzzy sets. Hur, Kang and Song [8-9] define and study the notion of intuitionistic fuzzy subring. Basnet [3] study the (α, β)-cut of intuitionistic fuzzy ideals of a ring. Sharma [16-17] study the translation of intuitionistic fuzzy subring and ideal. Here in this paper, we introduce the notion of intuitionistic fuzzy bi-ideal in a ring and study some of their properties.

2 Preliminaries

Throughout this paper, let R denote a ring unless other specified.

Definition (2.1)[3, 13] Let R be a ring. An IFS A = {< x, μA(x), νA(x) >: x ∈R} of R is said to be intuitionistic fuzzy subring of R (IFSR) of R if

(i) μA(x-y) ≥ Min{μA(x), μA(y)}
(ii) μA(xy) ≥ Min{μA(x), μA(y)}
(iii) νA(x-y) ≤ Max{νA(x), νA(y)}
(iv) νA(xy) ≤ Max{νA(x), νA(y)}, for all x, y ∈R

Definition (2.2)[13] An IFS A = {< x, μA(x), νA(x) >: x ∈R} of a ring R said to be
(a) intuitionistic fuzzy left ideal of R (IFLI) of R if
(i) μA(x-y) ≥ Min{μA(x), μA(y)}
(ii) μA(xy) ≥ μA(y)
(iii) νA(x-y) ≤ Max{νA(x), νA(y)}
(iv) νA(xy) ≤ νA(y), for all x, y ∈R

(b) intuitionistic fuzzy right ideal of R (IFRI) of R if
(i) μA(x-y) ≥ Min{μA(x), μA(y)}
(ii) μA(xy) ≥ μA(x)
(iii) νA(x-y) ≤ Max{νA(x), νA(y)}
(iv) νA(xy) ≤ νA(x), for all x, y ∈R

(c) intuitionistic fuzzy ideal of R (IFI) of R if
(i) μA(x-y) ≥ Min{μA(x), μA(y)}
(ii) μA(xy) ≥ Max{μA(x), μA(y)}
(iii) νA(x-y) ≤ Max{νA(x), νA(y)}
(iv) νA(xy) ≤ Min{νA(x), νA(y)}, for all x, y ∈R

Theorem (2.3)[13] If A = {< x, μA(x), νA(x) >: x ∈R} be IFSR of ring R , then
(i) μA(0) ≥ μA(x) and νA(0) ≤ νA(x)
(ii) μA(-x) = μA(x) and νA(-x) = νA(x), for all x, y ∈R
(iii) If R is ring with unity 1, then μA(1) ≤ μA(x) and νA(1) ≥ νA(x), for all x ∈R
Definition (2.4)[3] Let A be intuitionistic fuzzy set of a ring R. Then \((\alpha, \beta)\)-cut of A is a crisp subset \(C_{\alpha, \beta}(A)\) of the IFS A is given by \(C_{\alpha, \beta}(A) = \{x \in R : \mu_{A}(x) \geq \alpha, \upsilon_{A}(x) \leq \beta\}\), where \(\alpha, \beta \in [0,1]\) with \(\alpha + \beta \leq 1\).

Theorem (2.5)[3] Let A be an IFS of a ring R. Then A is IFSR (IFI) of R if and only if \(C_{\alpha, \beta}(A)\) is subring (ideal) of R, for all \(\alpha, \beta \in [0,1]\) with \(\alpha + \beta \leq 1\).

Definition (2.5) ([12]) A subring S of a ring R is called a bi-ideal of R if \(SRS \subseteq S\) holds, where SRS is the additive subgroup of R generated by the set of all elements of the form \(srs, s \in S\) and \(r \in R\).

Definition (2.6) ([5]) A non-empty fuzzy subset \(\mu\) of a ring R (i.e., \(\mu(x) \neq 0\) for some \(x \in R\)) is called an intuitionistic fuzzy bi-ideal of R if

(i) \(\mu(x-y) \geq \text{Min} \{\mu(x), \mu(y)\}\),
(ii) \(\mu(xy) \geq \text{Min} \{\mu(x), \mu(y)\}\), and
(iii) \(\mu(xyz) \geq \text{Min} \{\mu(x), \mu(y), \mu(z)\}\) for all \(x, y, z \in R\).

Definition (2.7) ([6]) A non-empty fuzzy subset \(\mu\) of a ring R (i.e., \(\mu(x) \neq 0\) for some \(x \in R\)) is called an anti intuitionistic fuzzy bi-ideal of R if

(i) \(\mu(x-y) \leq \text{Max} \{\mu(x), \mu(y)\}\),
(ii) \(\mu(xy) \leq \text{Max} \{\mu(x), \mu(y)\}\), and
(iii) \(\mu(xyz) \leq \text{Max} \{\mu(x), \mu(y), \mu(z)\}\) for all \(x, y, z \in R\).

3 Intuitionistic fuzzy bi-ideal of a ring

Definition (3.1) An IFS \(A = \{x, \mu_{A}(x), \upsilon_{A}(x) > x \in R\}\) of a ring R is said to be intuitionistic fuzzy bi-ideal of R (IFBI) of R if

(i) \(\mu_{A}(x-y) \geq \text{Min} \{\mu_{A}(x), \mu_{A}(y)\}\)
(ii) \(\upsilon_{A}(x-y) \leq \text{Max} \{\upsilon_{A}(x), \upsilon_{A}(y)\}\) for all \(x, y \in R\)
(iii) \(\mu_{A}(xy) \geq \text{Min} \{\mu_{A}(x), \mu_{A}(y)\}\)
(iv) \(\upsilon_{A}(xy) \leq \text{Max} \{\upsilon_{A}(x), \upsilon_{A}(y)\}\), for all \(x, y \in R\)
(v) \(\mu_{A}(xry) \geq \text{Min} \{\mu_{A}(x), \mu_{A}(y)\}\)
(vi) \(\upsilon_{A}(xry) \leq \text{Max} \{\upsilon_{A}(x), \upsilon_{A}(y)\}\), for all \(x, y \in R\).

Example 1. Let R be the ring of all 2x2 matrices over the ring of integers with respect to the matrix addition and multiplication. Let \(A = \{x, \mu_{A}(x), \upsilon_{A}(x) > x \in R\}\) be an intuitionistic fuzzy subset of R defined as follows:

\[
\mu_{A}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} 1 & \text{if } a = b = c = d = 0 \\
\frac{1}{2} & \text{if } a \neq 0, \text{ even and } b = c = d = 0 \\
\frac{1}{3} & \text{if } a \neq 0, \text{ odd and } b = c = d = 0 \\
0 & \text{in all other cases} \end{cases}
\]

Then A is a intuitionistic fuzzy bi-ideal of R.

Theorem (3.2) Every IFLI (IFRI) of a ring R is IFBI of R.

Proof. In view of (3.1), we need only to prove the condition (v) and (vi). Let A be IFLI of the ring R and \(x, y \in R\) and \(a \in A\). Then \(\mu_{A}(xay) \geq \mu_{A}(xay) \geq \mu_{A}(y) \geq \text{Min} \{\mu_{A}(x), \mu_{A}(y)\}\) and \(\upsilon_{A}(xay) \leq \upsilon_{A}(xay) \leq \upsilon_{A}(y) \leq \text{Max} \{\upsilon_{A}(x), \upsilon_{A}(y)\}\). Hence A is IFBI of R. The right case is proved in an analogous way.

Next, we give an example of a IFBI of the ring, which is neither IFLI nor IFRI.

Example 2. Consider the ring R of real numbers under usual addition and multiplication operations. Define the IFS \(A = \{x, \mu_{A}(x), \upsilon_{A}(x) > x \in R\}\) by

\[
\mu_{A}(x) = \begin{cases} 0.8 & \text{if } x \text{ is rational} \\
0.4 & \text{if } x \text{ is irrational} \end{cases}
\]

and

\[
\upsilon_{A}(x) = \begin{cases} 0.1 & \text{if } x \text{ is rational} \\
0.5 & \text{if } x \text{ is irrational} \end{cases}
\]

It is easy to check that A is neither a IFLI nor a IFRI of R. But A is a IFBI of R.

Theorem (3.3) Let \(A = \{x, \mu_{A}, \upsilon_{A}\}\) be IFBI of a field F. Then A is of the form
\[ \mu_A(x) = \begin{cases} \mu_A(1), & \text{if } x \neq 0 \\ \mu_A(0), & \text{if } x = 0 \end{cases} \]
and
\[ v_A(x) = \begin{cases} v_A(1), & \text{if } x \neq 0 \\ v_A(0), & \text{if } x = 0 \end{cases} \]

where \( \mu_A(1) \leq \mu_A(0) \) and \( v_A(1) \geq v_A(0) \).

**Proof.** Let \( A = \{ x, \mu_A(x), v_A(x) : x \in F \} \) be an IFBI of a field \( F \). Let \( 0 \neq x \in F \) be any element. Then
\[ \mu_A(x) = \mu_A(1 \cdot x) \geq \min \{ \mu_A(1), \mu_A(1) \} = \mu_A(1) = \mu_A(1.1) = \mu_A((x^X) \cdot (x^Y)) = \mu_A(x(x^Y) x) \geq \min \{ \mu_A(x), \mu_A(x) \} = \mu_A(x) \]
i.e. \( \mu_A(x) = \mu_A(1) \) and
\[ v_A(x) = v_A(1 \cdot x) \leq \max \{ v_A(1), v_A(1) \} = v_A(1) = v_A(1.1) = v_A((x^X) \cdot (x^Y)) = v_A(x(x^Y) x) \leq \max \{ v_A(x), v_A(x) \} = v_A(x) \]
i.e. \( v_A(x) = v_A(1) \)

**Corollary (3.4)** If \( A \) is IFBI of a field \( F \) such that \( \mu_A(0) = \mu_A(1) \) and \( v_A(0) = v_A(1) \), then \( A \) is constant.

**Theorem (3.5)** If \( A \) and \( B \) be two IFBI’s of a ring \( R \), then
\( A \cap B \) is IFBI of ring \( R \).

**Proof.** Let \( A = (\mu_A, v_A) \) and \( B = (\mu_B, v_B) \) be two IFBI’s of a ring \( R \). Let \( x, y \in A \cap B \) be any element. Then
\[ \mu_{A \cap B}(x-y) = \min \{ \mu_A(x-y), \mu_B(x-y) \} \]
\[ \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \}, \min \{ \mu_B(x), \mu_B(y) \} \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(y), \mu_B(y) \} \} \]
\[ = \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \]
Thus \( \mu_{A \cap B}(x-y) \geq \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \)

Similarly, we can show that
\[ v_{A \cap B}(x-y) \leq \max \{ v_{A \cap B}(x), v_{A \cap B}(y) \} \]

Also, \( \mu_{A \cap B}(x+y) = \min \{ \mu_A(x), \mu_B(x) \} \)
\[ \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \}, \min \{ \mu_B(x), \mu_B(y) \} \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(y), \mu_B(y) \} \} \]
\[ = \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \]
Thus \( \mu_{A \cap B}(x+y) \geq \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \)

Next, let \( x, y \in A \cap B \) and \( r \in R \) be any element, then
\[ \mu_{A \cap B}(xry) = \min \{ \mu_A(xry), \mu_B(xry) \} \]
\[ \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \}, \min \{ \mu_B(x), \mu_B(y) \} \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(y), \mu_B(y) \} \} \]
\[ = \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \]
Thus \( \mu_{A \cap B}(xry) \geq \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \)

Similarly, we can show that
\[ v_{A \cap B}(xry) \leq \max \{ v_{A \cap B}(x), v_{A \cap B}(y) \} \]

Hence \( A \cap B \) is IFBI of ring \( R \).

**Corollary (3.6)** Intersection of an arbitrary family of IFBI’s of a ring \( R \) is again an IFBI of \( R \).

**Theorem (3.7)** Let \( A \) be IFLI and \( B \) be IFRI of a ring \( R \), then \( A \cap B \) is IFBI of ring \( R \).

**Proof.** Let \( A = (\mu_A, v_A) \) and \( B = (\mu_B, v_B) \) be two IFBI’s of a ring \( R \). Let \( x, y \in A \cap B \) be any element. Then
\[ \mu_{A \cap B}(x-y) = \min \{ \mu_A(x-y), \mu_B(x-y) \} \]
\[ \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \}, \min \{ \mu_B(x), \mu_B(y) \} \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(y), \mu_B(y) \} \} \]
\[ = \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \]
Thus \( \mu_{A \cap B}(x-y) \geq \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \)

Similarly, we can show that
\[ v_{A \cap B}(x-y) \leq \max \{ v_{A \cap B}(x), v_{A \cap B}(y) \} \]

Also, \( \mu_{A \cap B}(xy) = \min \{ \mu_A(xy), \mu_B(xy) \} \)
\[ \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \}, \min \{ \mu_B(x), \mu_B(y) \} \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(y), \mu_B(y) \} \} \]
\[ = \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \]
Thus \( \mu_{A \cap B}(xy) \geq \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \)

Further, let \( x, y \in A \cap B \) and \( r \in R \), then
\[ \mu_{A \cap B}(xry) = \min \{ \mu_A(xry), \mu_B(xry) \} \]
\[ \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \}, \min \{ \mu_B(x), \mu_B(y) \} \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(y), \mu_B(y) \} \} \]
\[ = \min \{ \mu_{A \cap B}(x), \mu_{A \cap B}(y) \} \]
But \( \mu_A(xy) \geq \mu_A((xy)y) \geq \mu_A(y) \) and \( \mu_B(xy) \geq \mu_B(xry) \geq \mu_B(x) \) implies that
\[ \text{Min}\{\mu_A(x), \mu_B(x)\} \geq \text{Min}\{\mu_A(y), \mu_B(x)\} \quad \ldots \ldots \quad (5) \]

As \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \), so \( \mu_{A \cap B}(y) \leq \mu_A(y) \) and 
\( \mu_{A \cap B}(x) \leq \mu_B(x) \)

\[ \Rightarrow \text{Min}\{\mu_A(y), \mu_B(x)\} \geq \text{Min}\{\mu_{A \cap B}(y), \mu_{A \cap B}(x)\} \quad \ldots \ldots \quad (6) \]

From (4), (5) and (6), we get 
\[ \mu_{A \cap B}(xy) \geq \text{Min}\{\mu_{A \cap B}(y), \mu_{A \cap B}(x)\} \]

Similarly, we can show that 
\[ \nu_{A \cap B}(x) \leq \text{Max}\{\nu_{A \cap B}(y), \nu_{A \cap B}(x)\} \]

Hence \( A \cap B \) is IFBI of ring \( R \).

**Theorem (3.8)** Let \( A \) be IFS of a ring \( R \), then \( A \) is IFBI of \( R \) if and only if \( C_{\alpha, \beta}(A) \) is bi-ideal of \( R \), for all \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \), where \( \mu_A(0) \geq \alpha \) and \( \nu_A(0) \leq \beta \)

**Proof.** Firstly, let \( A \) be IFBI of a ring \( R \). Then by definition of \( (\alpha, \beta) \)-cut of \( A \), we have
\[ C_{\alpha, \beta}(A) = \{ x \in R : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \} \]

Since \( \mu_A(0) \geq \alpha \), \( \nu_A(0) \leq \beta \) \( \Rightarrow \) \( C_{\alpha, \beta}(A) \neq \emptyset \).

Let \( x, y \in C_{\alpha, \beta}(A) \) be any elements, then
\[ \mu_A(x) \geq \alpha, \quad \mu_A(y) \geq \alpha, \quad \nu_A(x) \leq \beta, \quad \nu_A(y) \leq \beta \]

\[ \Rightarrow \text{Min}\{\mu_A(x), \mu_A(y)\} \geq \alpha \quad \text{and} \quad \text{Max}\{\nu_A(x), \nu_A(y)\} \leq \beta \]

Now \( \mu_A(xy) \geq \text{Min}\{\mu_A(x), \mu_A(y)\} \geq \alpha \) and 
\[ \nu_A(xy) \leq \text{Max}\{\nu_A(x), \nu_A(y)\} \leq \beta \]

Also, \( \mu_A(xy) \geq \text{Min}\{\mu_A(x), \mu_A(y)\} \geq \alpha \) and 
\[ \nu_A(xy) \leq \text{Max}\{\nu_A(x), \nu_A(y)\} \leq \beta \]

\[ \Rightarrow x - y \in C_{\alpha, \beta}(A) \quad \text{and} \quad xy \in C_{\alpha, \beta}(A). \quad \text{Thus} \quad C_{\alpha, \beta}(A) \quad \text{is a subring of} \quad R. \]

Next, let \( x, y \in C_{\alpha, \beta}(A) \) and \( r \in R \) be any element. Then
\[ \mu_A(xr) \geq \text{Min}\{\mu_A(x), \mu_A(y)\} \geq \alpha \quad \text{and} \quad \nu_A(xr) \geq \text{Max}\{\mu_A(x), \mu_A(y)\} \leq \beta \]

\[ \Rightarrow xr \in C_{\alpha, \beta}(A). \quad \text{Thus} \quad C_{\alpha, \beta}(A)RC_{\alpha, \beta}(A) \subseteq C_{\alpha, \beta}(A) \]

Hence \( C_{\alpha, \beta}(A) \) is bi-ideal of \( R \).

**Conversely**, let \( C_{\alpha, \beta}(A) \) be bi-ideal of \( R \), for all \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \), where \( \mu_A(0) \geq \alpha \) and \( \nu_A(0) \leq \beta \).

Then \( C_{\alpha, \beta}(A) \) is a subring of \( R \) and \( C_{\alpha, \beta}(A)RC_{\alpha, \beta}(A) \subseteq C_{\alpha, \beta}(A) \).

This implies that \( A \) is intuitionistic fuzzy subring of \( R \) (by Theorem (2.5)).

Let \( x, y, r \in R \) and such that \( \mu_A(xr) < \text{Min}\{\mu_A(x), \mu_A(y)\} \)

Choose \( \alpha \in [0,1] \) such that \( \mu_A(xr) < \alpha < \text{Min}\{\mu_A(x), \mu_A(y)\} \),

this implies that \( \mu_A(x) \geq \alpha, \mu_A(y) \geq \alpha \Rightarrow \nu_A(x) \leq 1 - \mu_A(x) \leq 1 - \alpha \) and \( \nu_A(y) \leq 1 - \mu_A(y) \leq 1 - \alpha \).

Thus, \( x, y \in C_{\alpha, \alpha}(A) \) and so \( xy \in C_{\alpha, \alpha}(A) \), i.e.

\( \mu_A(xr) \geq \alpha \), a contradiction.

So, \( \mu_A(xr) \geq \text{Min}\{\mu_A(x), \mu_A(y)\} \). Similarly, we have 
\[ \nu_A(xr) \leq \text{Max}\{\nu_A(x), \nu_A(y)\} \]

Hence \( A \) is IFBI of the ring \( R \).

**Theorem (3.9)** If every bi-ideal of a ring \( R \) is a ideal of \( R \), then every IFBI of \( R \) is IFI of \( R \).

**Proof.** Let \( A \) be IFBI of \( R \). Then by theorem (3.8), \( C_{\alpha, \beta}(A) \) be bi-ideal of \( R \), for all \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \), which implies that \( C_{\alpha, \beta}(A) \) be ideal of \( R \), for all \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \) and so by theorem (2.5), \( A \) is IFI of \( R \).

### 4. Intuitionistic fuzzy magnified translation of bi-ideal of a ring

The notion of intuitionistic fuzzy magnified translation(IFMT) of intuitionistic fuzzy set has been defined and discussed by the first author in [16]. Here, in this section, we discuss the IFMT of IFBI of a ring and its homomorphic image.

**Definition (4.1)** Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy subset of \( X \) and \( \beta \in [0,1] \) and \( \alpha \in [0,1] \). Define the intuitionistic fuzzy subset of \( X \) by \( A_{\beta}^{\alpha} \) such that 
\[ A_{\beta}^{\alpha} = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \} \]

where \( \mu_A(x) = \mu_A(x) \) and \( \nu_A(x) = \nu_A(x) \). The intuitionistic fuzzy magnified translation (IFMT) \( T \) of \( A \) is an object of the form 
\[ T = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \} \]

where the functions \( \mu_A(x) = \mu_A(x) \) and \( \nu_A(x) = \nu_A(x) \) are defined as:
\[ \mu_T(x) = \mu_{T,\alpha}^A(x) = \mu_T(x) + \alpha, \quad \nu_T(x) = \nu_{T,\alpha}^A(x) = \nu_T(x) + \alpha \]

for all \( x \in X \).

**Example (4.2):** Let \( X = \{1, \omega, \omega^2\} \). Let \( A = \{<1, 0.3, 0.4>, <\omega, 0.1, 0.25>, <\omega^2, 0.5, 0.3>\} \) be an IFS of \( X \). Then \( 0.1 - \text{Sup} \{ \mu_A(x) + \nu_A(x): x \in X, 0 < \mu_A(x) + \nu_A(x) < 1 \} = [0, 0.2] \). Take \( \alpha = 0.1 \) and \( \beta = 0.2 \). Then IFMT of the IFS \( A \) is given by \( T = \{<1, 0.16, 0.18>, <\omega, 0.12, 0.15>, <\omega^2, 0.2, 0.16>\} \).

**Theorem (4.3):** Let \( T \) be an IFMT of an IFBI \( A \) of a ring \( R \), then \( T \) is also an IFBI of \( R \).

**Proof:** Assume that \( T \) is an IFMT of an IFBI \( A \) of a ring \( R \). Let \( x, y, z \in R \), we have

\[ \mu_T(x - y) = \beta \mu_A(x - y) + \alpha \]
\[ \geq \beta \text{Min} \{ \mu_A(x) , \mu_A(y) \} + \alpha \]
\[ = \text{Min} \{ \beta \mu_A(x) + \alpha , \beta \mu_A(y) + \alpha \} \]
\[ = \text{Min} \{ \mu_T(x) , \mu_T(y) \} \]

also
\[ \mu_T(xy) = \beta \mu_A(xy) + \alpha \]
\[ \geq \beta \text{Min} \{ \mu_A(x) , \mu_A(y) \} + \alpha \]
\[ = \text{Min} \{ \beta \mu_A(x) + \alpha , \beta \mu_A(y) + \alpha \} \]
\[ = \text{Min} \{ \mu_T(x) , \mu_T(y) \} \]

also
\[ \nu_T(x - y) = \beta \nu_A(x - y) + \alpha \]
\[ \leq \beta \text{Max} \{ \nu_A(x) , \nu_A(y) \} + \alpha \]
\[ = \text{Max} \{ \beta \nu_A(x) + \alpha , \beta \nu_A(y) + \alpha \} \]
\[ = \text{Max} \{ \nu_T(x) , \nu_T(y) \} \]

And
\[ \nu_T(xy) = \beta \nu_A(xy) + \alpha \]
\[ \leq \beta \text{Max} \{ \nu_A(x) , \nu_A(y) \} + \alpha \]
\[ = \text{Max} \{ \beta \nu_A(x) + \alpha , \beta \nu_A(y) + \alpha \} \]
\[ = \text{Max} \{ \nu_T(x) , \nu_T(y) \} \]

Hence \( T \) is also an intuitionistic fuzzy bi-ideal of \( R \).

**Example (4.4):** Let \( T \) be an IFMT of an IFBI \( A \) of a ring \( R \), then \( H = \{ x \in R : \mu_T(x) = \mu_T(0) \text{ and } \nu_T(x) = \nu_T(0) \} \) is bi-ideal of \( R \).

**Proof:** Easy exercise

**Example (4.5):** Let \( T \) be an IFMT of an IFBI \( A \) of a ring \( R \), then \( H = \{ x \in R : \mu_T(x) = \mu_T(0) \text{ and } \nu_T(x) = \nu_T(0) \} \) is a fuzzy bi-ideal of \( R \).

**Proof:** Easy exercise

**Example (4.6):** Let \( T \) be an IFMT of an IFBI \( A \) of a ring \( R \), then \( H = \{ x \in R : \mu_T(x) = \mu_T(0) \text{ and } \nu_T(x) = \nu_T(0) \} \) is an anti-fuzzy bi-ideal of \( R \).

**Proof:** Easy exercise.

**Proposition (4.7):** Let \( R \) and \( R' \) be any two rings. Then the homomorphic image of an IFMT of an IFBI of \( R \) is an IFBI of \( R' \).

**Proof:** Let \( R \) and \( R' \) be any two rings and \( f: R \to R' \) be a ring homomorphism.

Therefore
\[ f(x + y) = f(x) + f(y) \]
and
\[ f(xy) = f(x)f(y) \]
for all \( x \) and \( y \in R \).

Now, for \( f(x) \) and \( f(y) \) in \( R' \), we have
\[ \mu_V(f(x) - f(y)) \geq \mu_V[f(x - y)] \]
\[ \geq \mu_T(x - y) \]
\[ = \beta \mu_A(x - y) + \alpha \]
\[ \geq \beta \text{Min}\{ \mu_A(x), \mu_A(y) \} + \alpha \]
\[ = \text{Min}\{ \beta \mu_A(x) + \alpha, \beta \mu_A(y) + \alpha \} \]
\[ = \text{Min}\{ \mu_T(x), \mu_T(y) \} \]

Thus
\[ \mu_V[f(x - f(y))] \geq \text{Min}\{ \mu_V(f(x)), \mu_V(f(y)) \} \]

And
\[ \mu_V[f(x)f(y)] = \mu_V[f(xy)] \]
\[ \geq \mu_T(x \ y) \]
\[ = \beta \mu_A(x \ y) + \alpha \]
\[
\begin{align*}
\geq & \beta \min \{ \mu_\alpha(\alpha), \mu_\beta(\beta) \} + \alpha \\
= & \min \{ \beta \mu_\alpha(\alpha) + \alpha, \beta \mu_\beta(\beta) + \alpha \} \\
= & \min \{ \mu_\psi(f(\alpha), \mu_\nu(f(\beta)) \}
\end{align*}
\]

Thus \( \mu_\psi[f(\alpha)f(\beta)] \geq \min \{ \mu_\psi(f(\alpha)), \mu_\psi(f(\beta)) \} \)

Also \( \nu_\psi[f(\alpha)f(\beta)] = \psi[f(xy)] \)

\[
\begin{align*}
\leq & \nu_\psi(\alpha \beta) \\
= & \beta \nu_\psi(\alpha \beta) + \alpha \\
\leq & \beta \max \{ \nu_\psi(\alpha) + \alpha, \nu_\psi(\beta) + \alpha \} \\
= & \max \{ \beta \nu_\psi(\alpha) + \alpha, \beta \nu_\psi(\beta) + \alpha \} \\
= & \max \{ \nu_\psi(f(\alpha), \nu_\psi(f(\beta)) \}
\end{align*}
\]

Thus \( \nu_\psi[f(\alpha)f(\beta)] \leq \max \{ \nu_\psi(f(\alpha), \nu_\psi(f(\beta)) \} \)

And \( \mu_\psi[f(\alpha)f(\beta)f(\gamma)] = \mu_\psi[f(xy)] \)

\[
\begin{align*}
\geq & \mu_\psi(\alpha \beta) \\
= & \beta \mu_\psi(\alpha \beta) + \alpha \\
\geq & \min \{ \mu_\psi(\alpha), \mu_\psi(\beta) \} + \alpha \\
= & \min \{ \beta \mu_\psi(\alpha) + \alpha, \beta \mu_\psi(\beta) + \alpha \} \\
= & \min \{ \nu_\psi(f(\alpha), \nu_\psi(f(\beta)) \}
\end{align*}
\]

Thus \( \mu_\psi[f(\alpha)f(\beta)f(\gamma)] \geq \min \{ \mu_\psi(f(\alpha)), \mu_\psi(f(\beta)), \mu_\psi(f(\gamma)) \} \)

4 References


