Abstract—The paper presents a registration algorithm based on the Analytical Fourier-Mellin Transform (AFMT). In the Fourier Mellin registration algorithm, the estimation of the geometrical transformation is only done in the Fourier Transform phase domain which is known by containing the shape information. The proposed algorithm will be used to estimate homography since in the vicinity of key points, we can assume that geometrical transformation is simply a similarity. So, we compute AFMT at each key point location and we use the phase correlation to match features. Then, we estimate homography parameters with RANSAC algorithm. An archeological application will illustrate the efficiency of AFMT registration in image mosaicing context by the construction of Punic Panorama.

Keywords: Analytical Fourier-Mellin Transform (AFMT), Phase correlation, Mosaicing.

1. Introduction

Image registration is considered as an important research topic in the field of image processing, and it aims at data integration and finding the geometric transformation from a scene to the other. The automatic registration techniques overcome the difficulties of merging the images mentally by integrating them into a single representation. Generally speaking, registration methods can be reformulated by combining the following four components: a space of primitives, a research space, a similarity measure and a search strategy [5]. The selection of the similarity measure is closely linked to the selection of matching primitives and therefore to the space of primitives. Among the typical similarity measures, we can mention those related to the correlation, to the sum of absolute differences and to the phase correlation. In order to introduce the proposed invariant similarities registration algorithm based on AFMT, it is useful to recall some definitions and notations which are formulated in [1],[2],[19] and [20]. Thus, we denote by $L^1(G,\mu)$ the normed vector space of complex functions defined on a given group $G$.

$$f \in L^1(G,\mu) \iff \int_G |f(x)|d\mu(x) < \infty,$$  \hspace{1cm} (1)

$f$ represents the gray level of an image at the geometrical location $x$. The abstract harmonic analysis theory extends the classical Fourier Transform to the Fourier Transform on a given group $G$ with the following definition:

Definition: The Fourier Transform on a commutative group $G$ of a given function $f$ belonging to $L^1(G,\mu)$ is defined as:

$$\hat{f}(\lambda) = \int_G f(x)|T_\lambda(x)|^{-1}d\mu(x),$$  \hspace{1cm} (2)

Where $d\mu$ the normalized invariant is measure of $G$ and $T_\lambda(x)$ represents all irreducible and unitary representation of $G$. It is known that the direct of planar similarities forms
a group which is equivalent to the space of polar coordinates \((\mathbb{R}_+^*, S^1)\) which could be parameterized as follow:

\[
(\mathbb{R}_+^*, S^1) = \{(r, \theta)| r > 0, \text{and} 0 < \theta < 2\pi\} \tag{3}
\]

\(G\) forms a compact and commutative low under the following operation:

\[
(r, \theta)(r', \theta') = (rr', \theta + \theta'), \tag{4}
\]

Its unique normalized invariant measure (Haar Measure) can be written as:

\[
d\mu(r, \theta) = \frac{dr}{r} d\theta, \tag{5}
\]

Thus, the Fourier transform of \(f\) on \(G\) can be defined as:

\[
f(k, v) = M_f(k, v) = \int_{0}^{\infty} \int_{0}^{2\pi} f(r, \theta)e^{-ik\theta}r^v \frac{dr}{r} d\theta, \tag{6}
\]

for \(k \in \mathbb{Z}\) and \(v \in \mathbb{R}\). It is called in literature the Fourier Mellin Transform of the irradiance distribution \(f(r, \theta)\) in a two-dimensional image expressed in polar coordinates. The origin of the polar coordinates is usually taken in the image center of gravity in order to achieve invariance under translations. The integral (6) diverges in general, since the convergence is indeed under the assumption that \(f(r, \theta)\) is equivalent to \(Kr^{\alpha}\) \((\alpha > 0\) and \(K\) is a constant\)in a neighborhood of the origin. A rigorous approach has been introduced to solve the singularity at the origin of coordinates. Ghorbel in [3] suggested computing the Analytical prolongation of Fourier Mellin Transform (AFMT) which is defined for \(k \in \mathbb{Z}\) and \(v \in \mathbb{R}\) as:

\[
M_{f_{\sigma}}(k, v) = \int_{0}^{\infty} \int_{0}^{2\pi} f(r, \theta)e^{-ik\theta}r^{\sigma-iv} \frac{dr}{r} d\theta, \tag{7}
\]

where \(\sigma > 0\) is a fixed and strictly positive real number. AFMT can be seen as the Laplace transform on the planar similarity group. It gives a unique description and images can be retrieved with the inverse of AFMT. Here we recall

Fig. 1: (a) Reference Image (Mosaic from Bardo mesum, Tunisia); (b) Image transformed (Mosaic from Bardo mesum, Tunisia); (c) Log-polar sampling of the image (1.b); (d) Analytical Fourier Mellin transform of the image (1.b)
its expression:

\[ f_2(r, \theta) = \int_0^\infty \sum_z M_{f_2}(k, v, \rho - \sigma + iv, e^{ik\theta}) dv, \]  

(8)

This can be used for the fast computation of the G correlation. By considering the AFMT of two images represented by the two functions \( f_1 \) and \( f_2 \) for the same analytic prolongation denoted respectively by \( M_{f_1} \) and \( M_{f_2} \), and by assuming that the product of the two spectrum \( M_{f_1}, M_{f_2} \) is in \( L^1(\mathbb{Z} \times \mathbb{R}) \), we apply the Inverse of AFMT to this product, we obtain the following correlation (equation (9)):

\[ C_{TFMA}(\alpha, \beta) = \int_0^\infty \sum_z M_{f_1}(\alpha + iv, e^{i\beta}) dv, \]  

(9)

3. Phase correlation based on AFMT of images

From the result presented above, an algorithm is now derived for the rotation and scale estimation. So, we apply the log-polar sampling on the two images (figure (1.c)). We have the following relation.

\[ \forall(r, \theta) \in G, f_1(r, \theta) = f_2(\frac{r}{\rho}, \theta - \varphi), \]  

(10)

Where \( f_1 \) and \( f_2 \) are two objects that have the same shape. Shall we denote the AFMTs of these two images for the same analytical prolongation respectively by \( M_{f_1} \) and \( M_{f_2} \). Figure (1.d) presents the Analytical Fourier Mellin transform of the image in figure (1.b). Based on the shift theorem in the planar similarities group, we have:

\[ M_{f_1} = \rho^{-\alpha^{iv}}e^{-ik\varphi}M_{f_2}, \]  

(11)

The phase correlation of two objects \( f_1(r, \theta) \) and \( f_2(r, \theta) \) can be written as:

\[ C_{TFMA}(\rho, \varphi) = \int_0^\infty \sum_z \phi(k, v, \alpha^{iv}) e^{i\sigma} dv, \]  

(12)

with

\[ \phi(k, v) = \frac{|M_{f_1}(k, v)| |M_{f_2}(k, v)|}{|M_{f_1}(k, v)| |M_{f_1}(k, v)|} = \rho^{-\alpha^{iv}} e^{-k\varphi}. \]  

(13)

Then, we search \((\rho_0, \varphi_0)\) that maximize correlation function. We apply log-polar sampling on gray scale images. We estimate the \( M_{f_2}(k, v) \), then, we calculate \( C_{TFMA}(\rho, \varphi) \).

4. The importance of phase in parameters estimation

Extensive experiments have been accomplished in order to assess the performance of the proposed algorithm. This section compares and presents the results of the transformation parameters estimation with the two algorithms described above. From the figure (1.a), we generated 30 images with different rotation angles. Picture (3) presents the resulting rotation estimation with the two correlation algorithm based on AFMT. It is obvious that the correlation algorithm based on the phase of AFMT is more robust and accurate in the geometric parameters estimation. Since the conventional algorithm based on the AFMT uses the magnitude which is very attached to the noise. The proposed algorithm detects the shape information. If we try to describe the surface of correlation generated with the two algorithm, we find a too prominent peak in the phase correlation surface (figure 2.d). But, the peak is not apparent in the surface of AFMT correlation (figure 2.c). It can causes estimation errors. From the figure (1.a), we generated 30 images with different rotation angles. We add a gaussian noise with \((m=0, \sigma=0.01)\). Picture (4) presents the resulting rotation estimation with Phase correlation algorithm based on AFMT; It is obvious that the correlation algorithm based on the phase of AFMT is more robust against noise.

5. Experimental results and application on image mosaicing

A mosaic is a compound image built through properly composing a high numbers of images and transforming them onto a common reference plan according to some geometric model. As stated in [11], we can classify the methods of creating a mosaic of images into two categories: dioptric and catadioptric. In the dioptric methods, only refractive elements (lenses) are used while in the catadioptric methods, some reflective elements (mirrors) will be added. Among the dioptric methods, we can find sets of cameras, panoramic lenses [12], "fish-eyes" lenses [13], linear cameras [14] or more conventionally rotating cameras. In the catadioptric methods, we usually find a camera attached to a conical [15], spherical [16], parabolic [17] or double curvature mirror, or when multiple cameras attached to plane mirrors [11]. We will present in this paper a new dioptric technique that allows the creation of images mosaic. We assume that the camera rotates about its optical centre. So, the group of transformation may undergo is a special group of homographies [18]. 2D homography is expressed with 8 coefficients by equation (14) where \( \vec{x'} = (x', y', z') \) is the coordinate of an arbitrary point in image \( J \) and \( \vec{x} = (x, y, z) \) is the coordinate of an arbitrary point in image \( I \)

\[ \vec{x'} = H \vec{x} \]  

(14)

There are several ways to determine the matrix \( H \). It is possible to determine the coefficients of the transformation matrix from the intrinsic and extrinsic parameters of the two images. More generally, registration techniques are classified.
Fig. 2: (a) The reference Image; (b) Transformed Image; (c) Surface of correlation between the reference (2.a) and the transformed image (2.b); (d) Surface of phase correlation between the reference (2.a) and the transformed image (2.b); (e) Mosaic of the two images (2.a) and (2.b)

according to the pixels contribution of the image. Two approaches are considered: the feature-based methods and the area-based ones [5].

5.1 Feature based registration methods

The principle of Feature-based registration methods [5] consists generally in matching specific points or geometrical structures in order to have enough equations to solve the linear system and to determine the transformation between the two images. Feature-based registration methods are therefore based on this principle:
1. Detection of interest points in both images.
2. Matching points.
3. Optimization Algorithm.

The first step is to extract the features. At each feature location, a characteristic or descriptor is established. The matching requires having a local descriptor of texture or color among which we can name SIFT, local jet and the frequency invariants as the Gabor transform, wavelet transform and the Fourier-Mellin transform. Frequency descriptors are widely used as texture descriptors. These descriptors depend on the nature of the geometric transformation. The coefficients of the conventional Fourier transform can be used. In case there are a rotation and a scale factor, the Fourier-Mellin transform is more appropriate. In this paper, we propose a registration algorithm based on phase correlation in the Analytical Fourier-Mellin space. We will integrate the phase correlation algorithm in a Feature-based registration system to perform the matching between points of interest issued
Fig. 3: Rotation estimation with the AFMT correlation and AFMT phase correlation

Fig. 4: Rotation estimation with the AFMT correlation and AFMT phase correlation (noisy images)

5.2 Feature based registration methods

Our idea is to apply the algorithm described in the fourth section but locally in the vicinity of key points. The first step is to detect points of interest in both images using Harris algorithm then we estimates the correlation between all detected points in both images. We chose a match for each point of interest in the reference image; it is the point which gives the highest correlation peak. Now, we must define a subset of pairs of points, test several geometrical transformations and keep the best solution. This approach is based on the RANSAC algorithm (RANdom SAMple Consensus) presented by Martin A. Fischler and al in [6]. This algorithm was originally developed to solve problems related to mapping. The more detailed description may be found in [7], [8], [9], [10],[19]. In our case, the solution we are looking for is an homography expressed from eight independent parameters (equation 14). Therefore, we need 4 pairs of points to solve the linear system. The principle of the algorithm is as follow. From two sets of points, we define a random subset of four pairs of points. These four pairs of points allow us to calculate a transformation matrix which is applied to the image $J$. If this solution minimizes the cost function, it is preserved, otherwise it is rejected. In both cases, a new solution is calculated. When a number of iterations is reached, the best intermediate solution $H'$ is used to determine precisely the pairs of points set. That is to

from the model image and the transformed one.
say, the points for which the distance between \( x' \) and \( x.H \) is below a threshold.

\[
\| x' - x.H \| \leq \epsilon
\]  

(15)

6. Conclusion

By means of the phase correlation algorithm rotation and scale can be estimated. In this paper, we demonstrated using practical example that it is more efficient to use phase correlation instead of correlation. The phase correlation algorithm based on the AFMT of images estimates the transformation parameters on the image by passing to the Fourier domain and keeps all the information defining the pictures. Integrating this algorithm within a mosaicking framework proved the robustness of the algorithm and the accuracy of the estimated parameters. Our objective is to propose an automatic approach of image registration based on phase correlation algorithm and to create a framework which involves four steps:

1. Identify the domain of definition of the image as a homogenous space and the group acting on it.
2. Check whether a Fourier Transform exists for the acting group.
3. Compute the correlation from the phase of Fourier Transform.
4. Estimate the parameters of geometrical transformation.

References