Abstract — In this letter we investigate how to optimize the frequency discrimination of multi-tone signals based using the warped discrete Fourier transform (WDFT). Compared to a conventional DFT or FFT, which has a uniform frequency resolution across the entire baseband, the frequency resolution of the WDFT is non-linear and externally controlled. This feature can be used to overcome the multi-tone signal detection limitations of the DFT/FFT. The letter demonstrates that by intelligently controlling the frequency resolution of the WDFT, multi-tone signals can be more readily detected and classified. Furthermore, the WDFT can be built upon an FFT enabled framework, insuring high efficiency and bandwidths.

Keywords — Spectral analysis, Frequency discrimination, Warped discrete Fourier transform (WDFT), Spectral leakage

I. INTRODUCTION

Multi-tone signal detection and discrimination is a continuing signal processing problem. The applications include dual-tone multi-frequency (DTMF) systems, Doppler radar, electronic countermeasures, wireless communications, OFDM-based radar exciters, to name but a few. Traditional multi-tone detection systems are based on a filterbank architecture that use an array of product modulators to heterodyne signals down to DC, and then processes the down-converted array of signals with a bank of lowpass filters. The output of the filterbank is then processed using a suite of energy detection operations to detect the presence of tones and multiple tones [1]. The capability of such a system to isolate and detect multiple narrowband signals is predicated on the choice of initial modulating frequencies and post-processing algorithms. Other approaches to the problem are based on multiple signal classification (MUSIC) algorithms, least mean-square (LMS) estimators, and DFT derivatives such as Goertzel algorithm [2]. The approach taken in this paper is to explore the use of another DFT derivative called the warped discrete Fourier transform or WDFT [3, 4].

II. FFT - THE ENABLING TECHNOLOGY

The discrete Fourier transform (DFT) is indisputably an important signal analysis tool, finding applications in virtually all engineering and scientific endeavors. Generally, the preferred implementation of the DFT is the venerable Cooley-Tukey fast Fourier transform (FFT) algorithm. An $N$-point DFT $X[k]$, $0 \leq k \leq N-1$, of a length-$N$ time-series $x[n]$, $0 \leq n \leq N-1$, is defined by:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \ 0 \leq k \leq N-1.$$ (1)

For spectral analysis applications, the DFT provides a uniform frequency resolution $\Delta = 2\pi/N$ over the normalized baseband $\omega \in [-\pi, \pi]$. That is, the DFT’s frequency resolution $\Delta$ is uniform across the entire baseband. This fact historically has limited the role of the DFT in performing acoustic and modal (vibration) analysis, applications that prefer to interpret a signal spectrum using logarithmic (octave) frequency dispersions. Another application area in which a fixed frequency resolution is a limiting factor is multi-tone signal detection and classification. It is generally assumed that if two tones are separated by $1.6\Delta$ (1.6 harmonics) or less, then a uniformly windowed DFT/FFT cannot determine if one tone or multiple tones are present at a harmonic frequency due to spectral leakage, which obscures the spectral separation between adjacent DFT harmonics [4]. This problem is exacerbated when data widows are employed (e.g., Hamming window). This condition is illustrated in Fig. 1. In Fig. 1(a), two tones separated by one harmonic (i.e., $\Delta$) are transformed. The output spectrum is seen to consist of a single peak, losing the
identification of each individual input tone because their main lobes get closer and eventually overlap. In the other case reported by Fig. 1(b), the two tones being transformed are separated in frequency by two harmonics (i.e., $2\Delta$). The presence of two distinct tones is now self-evident.

III. THE WDFT

The WDFT is a derivative of the familiar DFT filterbank [6]. It differentiates itself from the standard DFT filterbank in that it contains an additional pre-processed stage. The WDFT can be developed in the context of multi-rate and polyphase signal processing theory [7]. A polyphase multi-rate filter architecture, shown in Fig. 2, was used to implement a WDFT [8]. In Fig. 2, the filter function $A(z)$ is as a pre-processing all-pass filter. For the case where $A(z) = 1$, the design degenerates to a traditional uniform DFT filterbank [6, 7]. For the case where $A(z) = 1$ and the polyphase terms $P_i(z) = 1$, the architecture shown in Fig. 2 becomes an $N$-point DFT.

Formally the $N$-point WDFT, reported in Fig. 2 for $P_i(z)=1$, is defined in terms of a DFT and pre-processing the all-pass filtered data, filtered by $A(z)$, where:

$$A(z) = \frac{-a + z^{-1}}{1 - az^{-1}}$$

where “$a$” is real and is called the warping control parameter. For stability reasons, “$a$” ranges between -1 and 1.

To motivate the behavior of a WDFT, consider the experiment reported in Fig. 3. The uniform resolution DFT case ($P_i(z) = 1$ and $a = 0$) is compared to the non-uniform resolution case ($P_i(z) = 1$ and $a = \pm0.3$). The ability of locally control the frequency resolution of the WDFT is clearly demonstrated. In addition, if a lowpass subband shaping filter polyphase filter ($P(z) = \Sigma z^iP_i(z)$) is employed (e.g., approximate ideal lowpass FIR), then additional control can be exercised over the shape and frequency selectivity of the WDFT spectrum.

Concentrating on the WDFT case where $P(z) = 1$, it may be recalled that the standard z-transform of an $N$-point input time series $x[n]$, namely

$$X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}.$$  (3)

The warping filter converts the input into:

$$\overline{X}(z) = \sum_{n=0}^{N-1} x[n]A(z)^n.$$  (4)

Recall that the conventional DFT of $x[n]$, namely $X[k]$, produces

- $\overline{\omega}$ corresponds to normalize baseband in $[0,\pi]$. 
- 16-channel uniform FFT spectrum (top left).
- 16-channel non-uniform FFT spectrum for $a=-0.3$ (top right).
- 16-channel non-uniform FFT spectrum for $a=0.3$ (bottom left).

Figure 3: WDFT Experiment.
a spectrum given by:

\[ X[k] = X(z) \mid_{z = e^{j2\pi k/N}} , \quad 0 \leq k \leq N - 1 \]  

(5)

where \( X[k] \) is evaluated at \( z = e^{j2\pi k/N} \), a point the uniformly resolved locations on the periphery of the unit circle in the z-domain. The WDFT coefficients, \( \tilde{X}[k] \), are similarly obtained by uniformly sampling \( \tilde{X}(z) \) at points on the unit circle in the z-domain, namely:

\[ \tilde{X}[k] = \tilde{X}(z) \mid_{z = e^{j2\pi k/N}} , \quad 0 \leq k \leq N - 1. \]  

(6)

The conventional uniform frequency resolution DFT, defined by \( z = e^{j\omega} \), has harmonic frequencies located at frequencies \( \omega = 2\pi k / N, \ k \in [0, \ldots, N - 1] \). The center frequencies of an N-point WDFT spectrum are located at the warped frequencies \( \tilde{\omega} \) where \( z = e^{j\tilde{\omega}} \), which are associated to \( \omega \) through the non-linear frequency warping relationship

\[
\tan \left( \frac{\tilde{\omega}}{2} \right) = \frac{1 - a}{1 + a} \tan \left( \frac{\omega}{2} \right) .
\]  

(7)

Equation (7) establishes a non-linear frequency warping relationship that is controlled by the real parameter “\( a \)”. A positive “\( a \)” provides higher frequency resolution on the high frequency region and a negative value of “\( a \)” increases frequency resolution in the low frequency region (see Figure 3).

The effect of the warping relationship is demonstrated in Fig. 4 which compares a DFT (\( a = 0 \)) to WDFTs for \( a = -0.071, a = -0.23 \) and \( a = -0.4 \) for the case where two tones are present separated by a single DFT harmonic. It is easily seen that by intelligently choosing the control parameter “\( a \)” the locally imposed frequency resolution can be expanded or contracted. To enhance the system’s frequency discrimination, the frequency resolution should be maximized in the local region containing the input signals. As such, an intelligent agent will need to assign the best warping parameter “\( a \)” strategy, one that concentrates the highest frequency resolution in the spectral region occupied by the multi-tone process.

The next section describes the outcome of a preliminary study that compares two criteria and two search algorithms and develops an “intelligent” frequency resolution discrimination policy that can be used to improve multi-tone detection.

IV. OPTIMIZATION OF FREQUENCY RESOLUTION

To optimize the choice of the warping parameter “\( a \)”, \( |a| < 1 \), an intelligent search algorithm or agent is needed. An initial search strategy is being evaluated and enabled using optimal single-variable search techniques, a Fibonacci search \(^1\) [9] and a modified Golden Section search \(^2\). The search process is expected to iterate over a range of values of “\( a \)” that places a high local frequency resolution in the region occupied by multi-tone activity. To find the best warping parameter “\( a \)”, two criteria of optimization and cost functionals have been singled out for focused attention. The search methods iteratively restrict and shift the search range so as to optimize spectral resolution within a convergent range. The direction of the search is decided by the value of the cost functional at two points in the range.

Two criteria studied to date are developed below.

A. Criterion #1

\[
\Phi_1(\bar{\omega}) = \max_i \left[ \tilde{X}[k] \right] - \sum_i \left[ \tilde{X}[k] \right]_{\sigma_j}
\]  

(8)

where \( \bar{\omega}_b \) is a frequency within the search interval and \( \Phi_1(\bar{\sigma}) \) is designed to reward a local concentration of spectral energy and penalize more sparsely populated section of the spectrum. The optimal operating point corresponds to a warping parameter “\( a \)” that maximizes the local spectral resolution in a region of signal activity.

B. Criterion #2:

\[
\Phi_2(\bar{\sigma}) = \sum \left[ \sigma - \tilde{X}[k] \right]
\]  

(9)

where \( \sigma \) is the threshold used to suppress leakage and \( \Phi_2(\bar{\sigma}) \) is designed to reward the local concentration of spectral energy.

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1 The Fibonacci search technique is a method of searching a sorted array using a divide and conquer algorithm that narrows down possible locations with the aid of Fibonacci numbers.

2 The Golden Section search is a technique for finding the extremum (minimum or maximum) of a unimodal function by successively narrowing the range of values inside which the extremum is known to exist.
V. RESULTS AND COMPARISON

The letter reports on a multi-tone signal discrimination study conducted using two search methods, namely a Fibonacci iterative methods that restrict and shift the searching range so as to determine an optimal operating point within a frequency range. Studies based on these criteria involved presenting to the “best” warping parameter. The comparison of results is shown in Table 1 and the evidence of this activity can be seen in Fig. 5. To compare the temporal efficiency of each case, Table 1 also shows elapsed time needed to execute a search using MATLAB. The two tones, separated by one harmonic, were unresolved with 64-point DFT but resolved with 512-point DFT at the expense of increased complexity (see Fig. 5 (a) and (b), respectively). In Fig. 5 (c)-(e), however, the two tones are seen to be present using 64-point WDFT. To calibrate the WDFT spectra, the locations of the actual two tones are also shown. Comparing the outcomes, a Fibonacci search was found to be the fastest and most effective in finding the “best” warping parameter using either search criteria. Criterion #1 resulted in frequency resolution with a bigger variation according to search methods, while Criterion #2 facilitated the optimization of the local resolution and identified the two tones closer to the actual locations of the tones.

VI. CONCLUSION

We have studied and shown that multi-tone processes can be successfully discriminated using the WDFT. The study demonstrates that the local frequency resolution can be selectively enhanced in regions of interest using the reported search techniques. Overall, the WDFT is found to be a very capable transform which can provide selective high spectral resolution at low complexity. Studies continue in this area, processing more complex signal cases.

Table 1
Comparison of the warping parameter

<table>
<thead>
<tr>
<th>Search Method</th>
<th>Criterion #1</th>
<th>Criterion #2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Warping parameter</td>
<td>Elapsed time (sec.)</td>
</tr>
<tr>
<td></td>
<td>$a = -0.1087$</td>
<td>$t = 0.749252s$</td>
</tr>
<tr>
<td></td>
<td>$a = -0.0721$</td>
<td>$t = 0.888743s$</td>
</tr>
<tr>
<td></td>
<td>$a = -0.0996$</td>
<td>$t = 0.736151s$</td>
</tr>
<tr>
<td></td>
<td>$a = -0.1381$</td>
<td>$t = 0.751035s$</td>
</tr>
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</table>

Figure 5. (a) Magnitude spectrum for two tones with (a) 64-point DFT ($a = 0$) and (b) 512-point DFT ($a = 0$) and magnitude spectrum for two tone detection with 64-point WDFT with (c) $a = -0.1087$, (d) $a = -0.0721$, (e) $a = -0.0996$ and (f) $a = -0.1381$.

REFERENCES