An Automated Deduction of the Independence of the Orthomodular Law from Ortholattice Theory

Jack K. Horner
PO Box 266
Los Alamos, New Mexico 87544 USA
email: jhorner@cybermesa.com

Abstract

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of quantum-mechanical behaviors and operations. In much the same way that the structure of conventional propositional logic is the logic of the description of the behavior of classical physical systems and is isomorphic to a Boolean lattice, so also the algebra, C(H), of closed linear subspaces of (equivalently, the system of linear operators on) a Hilbert space is a logic of the descriptions of the behavior of quantum mechanical systems and is a model of an ortholattice (OL). An OL can thus be thought of as a kind of “quantum logic” (QL). C(H) is also a model of an orthomodular lattice (OML), which is an OL conjoined with an orthomodularity axiom/law (OMA). The rationalization of the OMA as a claim proper to physics has proven problematic, motivating the question of whether the OMA is required in an adequate characterization of QL. Here, I use an automated deduction framework to show that the OMA is independent of the axioms of ortholattice theory. These results corroborate (and fix a minor defect in) previously published work characterizing the strength of the OMA, and demonstrate the utility of automated deduction in investigating quantum computing logic-optimization strategies.

Keywords: automated deduction, quantum computing, orthomodular lattice, Hilbert space

1.0 Introduction

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of the description of quantum-mechanical behaviors ([1], [17], [18], [20]). In much the same way that conventional propositional logic ([12]) is the logical structure of description of the behavior of classical physical systems (e.g., “measurements of the position and momentum of an electron are commutative”) and is isomorphic to a Boolean lattice ([10], [11], [19]), so also the algebra, C(H), of the closed linear subspaces of (equivalently, the system of linear operators on) a Hilbert space H ([1], [4], [6], [9], [13]) is a logic of the descriptions of the behavior of quantum mechanical systems (e.g., “the measurements of the position and momentum of an electron are not commutative”) and is a model ([10]) of an ortholattice (OL; [8]). An OL can thus be thought of as a kind of “quantum logic” (QL; [19]). C(H) is also a model of (i.e., isomorphic to a set of sentences which hold in) an orthomodular lattice (OML; [7], [8]), which is an OL conjoined with the orthomodularity axiom (OMA; see Figure 1).

The rationalization of the OMA as a claim proper to physics has proven problematic ([13], Section 5-6), motivating the question of whether the OMA is independent of the axioms of ortholattice theory.
Lattice axioms
\[
x = c(c(x)).
\]
\[
x v y = y v x.
\]
\[
(x v y) v z = x v (y v z).
\]
\[
(x ^ y) ^ z = x ^ (y ^ z).
\]
\[
x v (x ^ y) = x.
\]
\[
x ^ (x v y) = x.
\]
Ortholattice axioms
\[
c(x) ^ x = 0.
\]
\[
c(x) v x = 1.
\]
\[
c(c(x)) = x.
\]
\[
x ^ y = c(c(x) v c(y)).
\]
Orthomodularity law/axiom (OMA)
\[
x v (c(x) ^ (x v y)) = x v y.
\]
Definitions (useful, but not required)
\[
i1(x, y) = c(x) v (x ^ y).
\]
\[
i2(x, y) = i1(c(y), c(x)).
\]
\[
i3(x, y) = (c(x) ^ y) v (c(x) ^ c(y)) v i1(x, y).
\]
\[
i4(x, y) = i3(c(y), c(x)).
\]
\[
i5(x, y) = (x ^ y) v (c(x) ^ y) v (c(x) ^ c(y)).
\]
\[
le(x, y) = (x = (x ^ y)).
\]

where
\[
x, y \text{ are variables ranging over lattice nodes}
\]
\[
^ \text{is lattice meet}
\]
\[
v \text{is lattice join}
\]
\[
c(x) \text{ is the orthocomplement of } x
\]
\[
i1(x, y) \text{ means } x \rightarrow_1 y \text{ (Sasaki implication)}
\]
\[
i2(x, y) \text{ means } x \rightarrow_2 y \text{ (Dishkant implication)}
\]
\[
i3(x, y) \text{ means } x \rightarrow_3 y \text{ (Kalmbach implication)}
\]
\[
i4(x, y) \text{ means } x \rightarrow_4 y \text{ (non-tollens implication)}
\]
\[
i5(x, y) \text{ means } x \rightarrow_5 y \text{ (relevance implication)}
\]
\[
le(x, y) \text{ means } x \leq y
\]
\[
\text{= is equivalence ([12])}
\]
\[
1 \text{ is the maximum lattice element } (= x v c(x))
\]
\[
0 \text{ is the minimum lattice element } (= c(1))
\]

Figure 1. Axioms (and some useful definitions) of lattices, ortholattices, orthomodularity.

The axioms for lattices, ortholattices, and orthomodularity, are shown in Figure 1.

In a QL, the distributive law does not hold, because the distributive law implies commutativity of
(measurement-) propositions, and in general, the (measurement-)propositions of QL are not commutative. Roughly speaking, a QL can be thought of as a classical propositional logic (CL) in which the distribution law does not hold. Table 1 shows some further differences (and similarities) between classical propositional logic and quantum propositional logic.

Table 1. Quantum logic connective counterparts of classical logic connectives.

<table>
<thead>
<tr>
<th>Classical logic connective</th>
<th>Quantum propositional logic connective “counterpart”</th>
</tr>
</thead>
<tbody>
<tr>
<td>“x or y”</td>
<td>lattice join (x v y, or x ∪ y)</td>
</tr>
<tr>
<td>“x and y”</td>
<td>lattice meet(x ^ y, or x ∩ y)</td>
</tr>
<tr>
<td>“not-x”</td>
<td>orthocomplement (c(x), or x⊥)</td>
</tr>
<tr>
<td>“if x, then y” (implication)</td>
<td>x → y (i = [1</td>
</tr>
</tbody>
</table>

Note that there are five QL implications that satisfy the Birkhoff-von Neumann condition

\[(x \rightarrow_i y = 1) \iff (x \leq y), \quad i = 1, 2, \ldots, 5 \quad (\text{CBvN})\]

In classical propositional logic, there is only one implication, sometimes denoted “→₀”, that satisfies CBvN.

2.0 Method

The ortholattice and OMA axiomatizations of Megill, Pavičić, and Horner ([5], [14], [15], [16], [21]) were implemented in a mace4 ([2]) script ([3]) and then executed in that framework on a Dell Inspiron 545 with an Intel Core2 Quad CPU Q8200 (clocked @ 2.33 GHz) and 8.00 GB RAM, running under the Windows Vista Home Premium /Cygwin operating environment. More specifically, the script used in this work implements the derivation of a model (Figure 2) in which the ortholattice axioms conjoined with the negation of the OMA hold in at least one interpretation, demonstrating that the OMA is independent of the ortholattice axioms.

3.0 Results

Figure 2 shows a mace4 “domain size 6” model which demonstrates “the OMA is independent of the ortholattice axioms.”
assign(iterate_up_to,10).
set(Verbose).

formulas(theory).
x = c(c(x)).
x v y = y v x.
(x v y) v z = x v (y v z).
(x ^ y) ^ z = x ^ (y ^ z).
x v (x ^ y) = x.
x ^ (x v y) = x.
c(x) ^ x = 0.
c(x) v x = 1.
c(c(x)) = x.
x ^ y = c(c(x) v c(y)).
le(x,y) = x ^ y.
A v (c(A) ^ (A v B)) != A v B.
end_of_list.

interpretation( 6, [number=1, seconds=0], [ function(A, [ 2 ]),
function(B, [ 3 ]),
function(c(_,), [ 1, 0, 4, 5, 2, 3 ]),
function(^(_,_), [ 0, 0, 0, 0, 0, 0,
  0, 1, 2, 3, 4, 5,
  0, 2, 2, 2, 0, 0,
  0, 3, 3, 3, 0, 0,
  0, 4, 0, 0, 4, 5,
  0, 5, 0, 0, 5, 5 ]),
function(il(_,_), [ 1, 1, 1, 1, 1, 1,
  0, 1, 2, 3, 4, 5,
  4, 1, 1, 1, 4, 4,
  5, 1, 1, 1, 5, 5,
  2, 1, 2, 2, 1, 1,
  3, 1, 3, 3, 1, 1 ]),
function(v(_,_), [ 0, 1, 2, 3, 4, 5,
  1, 1, 1, 1, 1, 1,
  2, 1, 2, 3, 1, 1,]
Figure 2. A “domain size 6” mace4 model in which the axioms of an ortholattice and the negation of the OMA hold (thus demonstrating that the OMA is independent of the ortholattice axioms). Each text line of values in the rightmost set of square brackets are assignments of values of integers to lattice positions. Such an assignment of values to nodes is said to satisfy or be a model of the indicated function/relation if the function/relation holds within the lattice under the assignment to those positions. Details of the mace4 notation can be found in [2].

The time to produce the content of Figure 2 on the platform described in Section 2.0 was approximately 0.06 second.

4.0 Discussion

The results shown in Section 3.0 motivate at least three observations:

1. The OMA is independent of the ortholattice axioms.

2. The proof corrects a minor (right-associativity) defect in a previously published work ([21]) on the strength of the OMA.

3. A proof that the OMA is independent of the ortholattice axioms is implied by the proof in [22], which shows that the OMA is also independent of ortholattice axioms conjoined with the weak orthomodularity law.

5.0 Acknowledgements

This work benefited from discussions with Tom Oberdan, Frank Pecchioni, and Tony Pawlicki. For any infelicities that remain, I am solely responsible.

6.0 References


