The Parameterized Complexity of Perfect Code in Graphs without Small Cycles

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Abstract—We study the parameterized complexity of k-PERFECT CODE in graphs without small cycles. We show that k-PERFECT CODE is W[1]-hard in bipartite graphs and thus in graphs with girth 4. On the other hand, we show that k-PERFECT CODE admits a $k^2 + k$ kernel in graphs with girth $\geq 5$.

Keywords: parameterized complexity, perfect code

1. Introduction

Parameterized complexity is a powerful framework that deals with hard computational problems. A parameterized problem is a set of instances of the form $(x,k)$, where $x$ is the input instance and $k$ is a nonnegative integer called the parameter. A parameterized problem is said to be fixed parameter tractable if there is an algorithm that solves the problem in time $\text{poly}(k)|x|^{O(1)}$, where $\text{poly}$ is a computable function solely dependent on $k$, and $|x|$ is the size of the input instance. The kernelization of a parameterized problem is a reduction to a problem kernel, that is, to apply a polynomial-time algorithm to transform any input instance $(x,k)$ to an equivalent reduced instance $(x',k')$ with $k' \leq k$ and $|x'| \leq g(k)$ for some function $g$ solely dependent on $k$. A parameterized problem is fixed parameter tractable if and only if it is kernelizable. On the other hand, many fixed parameter intractable problems can be classified in a hierarchy of complexity classes $\text{W}[1] \subseteq \text{W}[2] \ldots \subseteq \text{W}[i]$. For example, $\text{k-INDEPENDENT SET}$ and $\text{k-CLIQUE}$ are known to be $\text{W}[1]$-complete and $\text{k-DOMINATING SET}$ is known to be $\text{W}[2]$-complete. We refer the readers to [1], [2] for more details.

Let $G = (V,E)$ be an undirected graph. For a vertex $v \in V$, let $N(v)$ and $N[v]$ be the open neighborhood and closed neighborhood of $v$, respectively. A perfect code in $G$ is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V$, there is exactly one vertex in $N[v] \cap D$.

Definition 1.1: Given an input graph $G$ and a positive integer $k$, the $k$-PERFECT CODE problem is to determine whether $G$ has a perfect code of size at most $k$.

In the literatures $k$-PERFECT CODE is also known as EFFICIENT DOMINATING SET, PERFECT DOMINATING SET, and INDEPENDENT PERFECT DOMINATING SET. It is a well-known NP-complete problem. Its computational complexity in various classes of graphs has been extensively studied. See Lu and Tang [3] for an overview. In terms of parameterized complexity, $k$-PERFECT CODE is known to be $\text{W}[1]$-complete [4], [5] in general graphs. Guo and Niedermeier [6] showed that $k$-PERFECT CODE is fixed parameter tractable in planar graphs by giving a $84k$ kernel. Dawar and Kreutzer [7] showed that it is fixed parameter tractable in effectively nowhere-dense classes of graphs.

The girth of a graph is the length of the shortest cycle contained in the graph. In this paper we study the parameterized complexity of $k$-PERFECT CODE in graphs with certain girths, i.e., graphs without small cycles. The parameterized complexity of several related problems, including $\text{k-DOMINATING SET}$ and $\text{k-INDEPENDENT SET}$ [8], and $\text{k-CONNECTED DOMINATING SET}$ [9] in graphs without small cycles has been studied. In this paper we show that $k$-PERFECT CODE is $\text{W}[1]$-hard in bipartite graphs, and thus in triangle-free graphs or graphs with girth 4. Then we show that $k$-PERFECT CODE admits a $k^2 + k$ kernel in graphs with girth $\geq 5$ and is therefore fixed parameter tractable.

2. Main results

2.1 Bipartite Graphs

To show the $\text{W}[1]$-hardness of $k$-PERFECT CODE in bipartite graphs, we give a reduction from the problem $\text{k-MULTICOLORED CLIQUE}$: Given a graph $G = (V,E)$ and a vertex-coloring $\kappa : V \to \{1,2,\ldots,k\}$, decide whether $G$ has a clique of $k$ vertices containing exactly one vertex of each color. $\text{k-MULTICOLORED CLIQUE}$ is known to be $\text{W}[1]$-complete by Fellows et al. [10].

Theorem 2.1: $k$-PERFECT CODE is $\text{W}[1]$-complete in bipartite graphs.

Proof: Let $(G = (V,E),\kappa)$ be an input instance of $\text{k-MULTICOLORED CLIQUE}$. For each color $i$, $1 \leq i \leq k$, let $V_i$ be the set of vertices in $G$ with color $i$. Let $n_i$ be the number of vertices in $V_i$. Without loss of generality, we assume that $n_i > 1$ for all $i$. We fixed an ordering of the vertices in each $V_i$. To simplify the presentation, we abuse notations here: for two vertices $u,v \in V_i$, $u > v$ means $u$ is in front of $v$ with respect to the fixed ordering. Without loss of generality, we also assume that no edge in $G$ connects two vertices of the same color. For any two colors $i$ and $j$, $1 \leq i < j \leq k$, let $E_{ij}$ be the set of edges in $G$ that connect
vertices in \( V_i \) and \( V_j \). Let \( m_{ij} \) be the number of edges in \( E_{ij} \).

We construct a graph \( G' = (V', E') \). The vertex set \( V' \) is a union of the following sets of vertices:

\[
S_1 = \{ a[i, v], b[i, v] \mid 1 \leq i \leq k, v \in V_i \} \\
\cup \{ x[i] \mid 1 \leq i \leq k \}
\]

\[
S_2 = \{ c[i, j, e], d[i, j, e] \mid 1 \leq i < j \leq k, e \in E_{ij} \} \\
\cup \{ y[i, j] \mid 1 \leq i < j \leq k \}
\]

\[
S_3 = \{ f[i, j, v] \mid 1 \leq i < j \leq k, v \in V_j \}
\]

\[
S_4 = \{ g[i, j, v] \mid 1 \leq i < j \leq k, v \in V_j \}
\]

The edge set \( E' \) is a union of the following set of edges:

\[
E_1 = \{ (a[i, v_1], b[i, v_2]) \mid 1 \leq i \leq k, v_1, v_2 \in V_i \} \\
\text{and } v_1 \neq v_2
\]

\[
E_2 = \{ (x[i], b[i, v]) \mid 1 \leq i \leq k, v \in V_i \}
\]

\[
E_3 = \{ (c[i, j, e_1], d[i, j, e_2]) \mid 1 \leq i < j \leq k, e_1, e_2 \in E_{ij} \}
\]

\[
E_4 = \{ (g[i, j], c[i, j, e]) \mid 1 \leq i < j \leq k, e \in E_{ij} \}
\]

\[
E_5 = \{ (b[i, v_1], f[i, j, v_2]) \mid 1 \leq i < j \leq k, v_1, v_2 \in V_i \text{ and } v_1 \geq v_2 \}
\]

\[
E_6 = \{ (b[j, v_1], g[i, j, v_2]) \mid 1 \leq i < j \leq k, v_1, v_2 \in V_j \text{ and } v_1 \geq v_2 \}
\]

\[
E_7 = \{ (c[i, j, e_1], f[i, j, v]) \mid 1 \leq i < j \leq k, e = (v_1, v_2) \in E_{ij}, v_1, v_2 \in V_i \text{ and } v_1 < v_2 \}
\]

\[
E_8 = \{ (c[i, j, e_1], g[i, j, v]) \mid 1 \leq i < j \leq k, e = (v_1, v_2) \in E_{ij}, v_2, v_1 \in V_j \text{ and } v_2 < v_1 \}
\]

Informally speaking, for each \( V_i \), \( 1 \leq i \leq k \), we construct a vertex selection gadget that contains \( 2n_i + 1 \) vertices. For each \( v \in V_i \), there are two vertices \( a[i, v] \) and \( b[i, v] \). For two vertices \( a[i, v_1] \) and \( b[i, v_2] \), \( v_1, v_2 \in V_i \), they are adjacent if and only if \( v_1 \neq v_2 \). There is a dummy vertex \( x[i] \) which is adjacent to all vertices \( \{ a[i, v] \mid v \in V_i \} \) and none of the vertices \( \{ a[i, v] \mid v \notin V_i \} \). Then, for each edge \( E_{ij} \), \( 1 \leq i < j \leq k \), we construct an edge selection gadget that contains \( 2m_{ij} + 1 \). For each edge \( e \in E_{ij} \), there are two vertices \( c[i, j, e] \) and \( d[i, j, e] \). There is also a dummy vertex \( y[i, j] \). They are connected in a similar fashion as in the vertex selection gadget. Finally, for each pair of colors \( i \) and \( j \) with \( 1 \leq i < j \leq k \), we also construct a validation gadget that contains \( n_i + n_j \) vertices, namely \( \{ f[i, j, v] \mid v \in V_j \} \) and \( \{ g[i, j, v] \mid v \in V_j \} \). The vertices in the validation gadget are not adjacent to each other, instead they are adjacent to vertices in the vertex selection gadgets for \( V_i \) and \( V_j \), and the edge selection gadget for \( E_{ij} \). For a vertex \( v_1 \in V_i \), the corresponding vertex \( b[i, v_1] \) is adjacent to \( f[i, j, v_2] \) for all \( v_2 \in V_j \) such that \( v_1 \geq v_2 \) with respect to the fixed vertex ordering of \( V_i \). Similarly, for a vertex \( v_1 \in V_j \), the corresponding vertex \( b[j, v_1] \) is adjacent to \( g[i, j, v_2] \) for all \( v_2 \in V_j \) such that \( v_1 \geq v_2 \) with respect to the fixed vertex ordering of \( V_j \). On the other hand, for an edge \( e = (v_1, v_2) \in E_{ij} \) with \( v_1 \in V_i \) and \( v_2 \in V_j \), the corresponding vertex \( c[i, j, e] \) in the edge selection gadget is adjacent to \( f[i, j, v] \) for all \( v \in V_j \) such that \( v > v_1 \), and to \( g[i, j, v] \) for all \( v \in V_j \) such that \( v > v_2 \). See Figure 1 for an illustration of the construction. Clearly \( G' \) is a bipartite graph.

![Fig. 1: A partial illustration of the construction of \( G' \).](image)

**Lemma 2.2:** \( G \) has a \( k \)-multicolored clique if and only if \( G' \) has a perfect code of size \( k' = 2k + 2k(k-1)/2 \).

**Proof:** For the direct implication, suppose \( G = (V, E) \) has a \( k \)-multicolored clique \( K \subseteq V \) such that \( K = \{ v_i \mid 1 \leq i \leq k, v_i \in V_i \} \), then it is easy to verify that the following set \( D \) of vertices in \( V' \) is a perfect code of size \( k' \) for \( G' \):

\[
D = \{ a[i, v_1], b[i, v_2] \mid v_i \in K \} \cup \{ c[i, j, e] \mid d[i, j, e] \mid 1 \leq i < j \leq k, e \in E_{ij} \}
\]

For the reverse implication, suppose \( D \) is a perfect code of size \( k' \) for \( G' \). First observe that the dummy vertex \( x[i] \) in the vertex selection gadget for \( V_i \) cannot be in \( D \) since otherwise vertices \( \{ a[i, v] \mid v \in V_i \} \) cannot be dominated. To dominate \( x[i] \), \( D \) must contain exactly one vertex from the set \( \{ b[i, v] \mid v \in V_i \} \). Let \( b[i, v] \) be such a vertex, \( b[i, v] \) dominates all vertices \( \{ a[i, v] \mid v \in V_i \} \) except \( a[i, v] \), this implies that \( a[i, v] \) must also be in \( D \). By this argument, we see that \( D \) must contain exactly two vertices from each vertex selection gadget and each edge selection gadget. In another word, the following \( 2k + 2k(k-1)/2 \) vertices must be in \( D \):

\[
\{ a[i, v_1], b[i, v_2] \mid 1 \leq i \leq k, v_i \in V_i \} \cup \{ c[i, j, e_1], d[i, j, e_2] \mid 1 \leq i < j \leq k, e \in E_{ij} \}
\]
So no vertex from the validation gadgets will be in $D$.

Let $b[i, v_i]$ and $b[j, v_j]$ be the two vertices in $D$. We see that $b[i, v_i]$ dominates vertices $f[i, j, v]$ for all $v \leq v_i$ in $V_i$ and $b[j, v_j]$ dominates vertices $g[i, j, v]$ for all $v \leq v_j$ in $V_j$. By the construction of $G'$, to perfectly dominate the rest of the validation vertices, $f[i, j, v]$ for all $v > v_i$ and $g[i, j, v]$ for all $v > v_j$, the vertex $c[i, j, e]$ must be in $D$ where $e = (v_i, v_j) \in E_{ij}$. Conversely, if $c[i, j, e]$ with $e = (v_i, v_j) \in E_{ij}$ is a vertex in $D$, $c[i, j, e]$ dominates $f[i, j, v]$ for all $v < v_i$ in $V_i$ and $g[i, j, v]$ for all $v < v_j$ in $V_j$, to perfectly dominate the rest of validation vertices, $b[i, v_i]$ and $b[j, v_j]$ must be in $D$. Therefore the set $\{v_i \mid b[i, v_i] \in D\}$ is a $k$-multicolored clique in $G$.

\[ 2.2 \text{ Graphs with girth } \geq 5 \]

Let $G = (V, E)$ be a graph with girth $\geq 5$. To obtain a $k^2 + k$ kernel, we only need the following simple reduction rule.

**Reduction Rule 1:** If a vertex $v \in V$ has degree $> k$, then remove $v$ and all vertices adjacent to $v$ from $G$ and decrease $k$ by 1.

**Lemma 2.3:** Reduction Rule 1 is correct.

**Proof:** Let $v \in G$ be a vertex with degree $> k$. We claim that if $G$ has a set $S$ which is a perfect code of size at most $k$, then $v$ must be in $S$. Suppose this is not true and $v \notin S$. Let $w_1, w_2, \ldots, w_l$ be the neighbors of $v$ with $l > k$. Since $v$ is not in $S$, exactly one of $v$'s neighbors must be in $S$. Without loss of generality, let $w_1$ be the vertex that is in $S$. $w_1$ is not adjacent to any $w_i$ for $1 < i \leq l$, since otherwise $v, w_1, w_i$ forms a triangle in $G$. Therefore $w_2, \ldots, w_l$ have to be dominated by vertices in $S$ other than $w_1$. We claim that any vertex $s \in S$ with $s \neq w_1$ can be adjacent to only one $w_i$. Since this is not true, i.e., there is a vertex $s \in S$ such that $(s, w_i), (s, w_j) \in E$ for $1 < i, j \leq l$, then $v, w_i, s, w_j$ forms a 4-cycle, contradicting that $G$ has girth $\geq 5$. Therefore $S$ contains $w_1$ and $l - 1$ more vertices, one for dominating each $w_i$ ($1 < i \leq l$), this makes $|S| \geq l$, contradicting the assumption that $|S| \leq k$.

Let $G'$ be the reduced graph after Reduction Rule 1. Clearly any vertex in $G'$ has degree at most $k$. Suppose $G'$ has a perfect code $S$ of size $k$, any vertex in $G'$ is either in $S$ or dominated by a vertex in $S$. Since each vertex in $S$ can dominate at most $k$ other vertices in $G'$, the size of $G' - S$ is at most $k^2$ and thus $G'$ has at most $k^2 + k$ vertices.

**Theorem 2.4:** $k$-PERFECT CODE admits a $k^2 + k$ kernel in graphs with girth $\geq 5$.

**References**


