An Automated Deduction of the Equivalence of Symmetry of Commutativity and the Orthomodular Law in Quantum Logic

Jack K. Horner
P. O. Box 266
Los Alamos, New Mexico 87544 USA
e-mail: jhorner@cybermesa.com

Abstract

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of quantum-mechanical behaviors and operations. In much the same way that the structure of conventional propositional (Boolean) logic (BL) is the logic of the description of the behavior of classical physical systems and is isomorphic to a Boolean algebra (BA), so also the algebra, C(H), of closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space is a logic of the descriptions of the behavior of quantum mechanical systems and is a model of an ortholattice (OL). An OL can thus be thought of as a kind of “quantum logic” (QL). C(H) is also a model of an orthomodular lattice, which is an OL conjoined with the orthomodularity axiom (OMA). The rationalization of the OMA as a claim proper to physics has proven problematic, motivating the question of whether the OMA and its equivalents are required in an adequate characterization of QL. Because the propositions of a QL are not in general commutative, quantum logicians have paid much attention to “quasi”-commutative theorems, one of the better known of which is the symmetry of commutativity theorem (SoCT), which states that commutativity is symmetric in an orthomodular lattice. Here I provide automated deductions showing that the SoCT and the OMA, in the context of a QL, are equivalent. The proofs appear to be novel.

Keywords: automated deduction, quantum computing, orthomodular lattice, commutativity, Hilbert space

1.0 Introduction

The optimization of quantum computing circuitry and compilers at some level must be expressed in terms of the description of quantum-mechanical behaviors ([1], [17], [18], [20]). In much the same way that conventional propositional (Boolean) logic (BL;[12]) is the logical structure of the behavior of classical physical systems (e.g., “the measurements of the position and momentum of particle P are commutative”) and is isomorphic to a Boolean lattice ([10], [11], [19]), so also the algebra, C(H), of the closed linear subspaces of (equivalently, the system of linear operators on (observables in)) a Hilbert space H ([1], [4], [6], [9], [13]) is a logic of the descriptions of the behavior of quantum mechanical systems (e.g., “the measurements of the position and momentum of particle P are not commutative”) and is a model ([10]) of an ortholattice (OL; [8]). An OL can thus be thought of as a kind of “quantum logic” (QL; [19]). C(H) is also a model of (i.e., isomorphic to a set of sentences which hold in) an orthomodular lattice (OML; [7], [8]), which is an OL conjoined with the orthomodularity axiom (OMA; see Figure 1). The rationalization of the OMA as a
claim proper to physics has proven problematic ([13], Section 5-6),

<table>
<thead>
<tr>
<th>Lattice axioms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = c(c(x))$ (AxLat1)</td>
</tr>
<tr>
<td>$x \lor y = y \lor x$ (AxLat2)</td>
</tr>
<tr>
<td>$(x \lor y) \lor z = x \lor (y \lor z)$ (AxLat3)</td>
</tr>
<tr>
<td>$(x \lor y) \land z = x \land (y \land z)$ (AxLat4)</td>
</tr>
<tr>
<td>$x \lor (x \land y) = x$ (AxLat5)</td>
</tr>
<tr>
<td>$x \land (x \lor y) = x$ (AxLat6)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ortholattice axioms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(x) \land x = 0$ (AxOL1)</td>
</tr>
<tr>
<td>$c(x) \lor x = 1$ (AxOL2)</td>
</tr>
<tr>
<td>$x \land y = c(c(x) \lor c(y))$ (AxOL3)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Orthomodularity axiom (aka Orthomodularity Law)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y \lor (c(y) \land (x \lor y)) = x \lor y$ (OMA)</td>
</tr>
</tbody>
</table>

where
- $x$, $y$ are variables ranging over lattice nodes
- $\land$ is lattice meet
- $\lor$ is lattice join
- $c(x)$ is the orthocomplement of $x$
- $=$ is equivalence ([12])
- $1$ is the maximum lattice element ($= x \lor c(x)$)
- $0$ is the minimum lattice element ($= c(1)$)

**Figure 1.** Lattice, ortholattice, and orthomodularity axioms.

Because of the fundamental role that non-commutativity plays in QL, quantum logicians have paid much attention to "quasi"-commutative theorems, which help to ground a large class of equivalence representations in quantum logic, and are thus of potential interest in optimizing quantum compiler and circuit design. Among the better known of the quasi-commutative theorems is the symmetry of commutativity theorem (SoCT; [7],[8]) shown is Figure 2

If $x$ and $y$ are elements of an orthomodular lattice,

$$xCy \leftrightarrow yCx$$

where $xCy$ means "$x$ commutes with $y"$, defined as
xCy <-> (x = ((x ^ y) v (x ^ c(y))))   

<-> means "if and only if"

Figure 2. The SoCT ([7],[8]).

Informally stated, the SoCT says that commutativity is symmetric in an orthomodular lattice. It turns out, as subsequent sections of this paper show, that the SoCT is equivalent to the OMA, in the sense that the axioms of an ortholattice, together with the SoCT, imply the OMA, and the axioms of an orthomodular lattice imply the SoCT.

2.0 Method

The OML and OL axiomatizations of Megill, Pavičić, and Horner ([5], [14], [15], [16], [21], [22]) were implemented in a prover9 ([2]) script configured to derive the equivalence of the SoCT and the OMA, then executed in that framework on a Dell Inspiron 545 with an Intel Core2 Quad CPU Q8200 (clocked @ 2.33 GHz) and 8.00 GB RAM, running under the Windows Vista Home Premium (SP2)/Cygwin operating environment.

3.0 Results

To prove the equivalence of the symmetry of commutativity with the Orthomodular Law, it suffices to prove the propositions shown in Sections 3.1, 3.2, and 3.3.

3.1 Proof of 'xCy ⇒ yCx' in an orthomodular lattice

Figure 3.1.1 shows the proof of proposition 'xCy ⇒ yCx' produced by [3] on the platform described in Section 2.0.
31 \[C(x, y) \mid c(c(x) v y) v c(c(x) v y) = x.\]
32 \[\text{copy}(30), \text{rewrite}([16(2), 16(7), 7(8), 8(9)])\].
33 \[C(x, y) \mid c(c(x) v y) v c(c(x) v y) = x.\]
34 \[\text{copy}(32), \text{rewrite}([16(2), 16(7), 7(8), 8(9)])\].
35 \[-C(c_2, c_1) \# \text{label("Commutativity is symmetric in an orthomodular lattice")}. \] \[\text{deny}(3)\].
36 \[C(c_1, c_2) \# \text{label("Commutativity is symmetric in an orthomodular lattice")} \# \text{answer("Commutativity is symmetric in an orthomodular lattice")}. \] \[\text{deny}(3)\].
37 \[c(1) = 0. \] \[\text{back_rewrite}(13), \text{rewrite}([16(2), 7(2), 15(2)])\].
38 \[c(c(x) v c(x v y)) = x. \] \[\text{back_rewrite}(12), \text{rewrite}([16(1)])\].
39 \[x v c(c(x) v c(y)) = x. \] \[\text{back_rewrite}(11), \text{rewrite}([16(1)])\].
40 \[x v (y v z) = y v (x v z). \] \[\text{para}(8(a,1), 9(a,1,1)), \text{rewrite}([9(2)])\].
42 \[x v (c(x) v y) = 1 v y. \] \[\text{para}(15(a,1), 9(a,1)), \text{flip(a)}\].
43 \[x v (y v c(x v y)) = 1. \] \[\text{para}(15(a,1), 9(a,1)), \text{flip(a)}\].
44 \[x v c(x v c(x v c(y))) = y v x. \] \[\text{para}(8(a,1), 18(a,1,2,1,2))\].
45 \[x v (c(x v c(y v x)) v z) = y v (x v z). \] \[\text{para}(18(a,1), 9(a,1,1)), \text{rewrite}([9(2)])\].
47 \[x v c(x v c(y v (z v x))) = y v (z v x). \] \[\text{para}(9(a,1), 18(a,1,2,1,1)), \text{rewrite}([9(8)])\].
50 \[C(c(x), y) \mid c(c(x) v y) v c(c(y) v c(x)) = c(x).\]
53 \[C(x, y) \mid c(c(x) v y) v c(c(y) v c(x)) = x.\]
58 \[c(c_2 v c(c_1)) v c(c(c_1) v c(c_2)) = c_1. \] \[\text{hyper}(31,a,34,a), \text{rewrite}([8(4)])\].
59 \[c(c_1 v c(c_2)) v c(c(c_1) v c(c_2)) = c_2 \# \text{answer("Commutativity is symmetric in an orthomodular lattice")}. \] \[\text{ur}(33,a,35,a), \text{rewrite}([8(4), 8(10)])\].
68 \[c(0 v c(x)) = x. \] \[\text{para}(15(a,1), 37(a,1,1,2,1)), \text{rewrite}([36(3), 8(3)])\].
71 \[C(x, x v y) \mid c(1 v y) v x \neq x. \] \[\text{para}(37(a,1), 33(b,1,2)), \text{rewrite}([40(5), 42(5)])\].
72 \[1 v x = 1. \] \[\text{para}(36(a,1), 37(a,1,1,1)), \text{rewrite}([68(6)])\].
74 \[C(x, 1) \mid x v 0 \neq x. \] \[\text{back_rewrite(63), \text{rewrite}([68(6), 72(5), 36(4)])\].
76 \[C(x, x v y) \mid 0 v x = x. \] \[\text{para}(36(a,1), 37(a,1,1,2)), \text{rewrite}([36(3), 8(3)])\].
81 \[C(x, x v y) \mid c(c(x) v y) = x. \] \[\text{para}(7(a,1), 38(a,1,2,1,2))\].
85 \[x v 0 = x. \] \[\text{para}(15(a,1), 38(a,1,2,1)), \text{rewrite}([36(2)])\].
86 \[x v c(y v c(x)) = c(x). \] \[\text{para}(7(a,1), 86(a,1,2,1,2))\].
93 \[C(x, 1) \mid c(0 v c(x)) v c(1 v c(x)) = c(x).\]
95 \[0 v x = x. \] \[\text{para}(15(a,1), 38(a,1,2,1)), \text{rewrite}([8(2)])\].
107 \[C(x, 1). \] \[\text{back_rewrite(74), \text{rewrite}([85(4)])}, xx(b)\].
125 \[C(x, x v y) \mid x v (c(c(x) v y) v z) = x v z. \] \[\text{para}(8(a,1), 18(a,1,2,1,2)), \text{rewrite}([8(26), 93(27)])\].
137 \[C(x, c(x) v y) \mid c(c(x) v z) v x = x. \] \[\text{para}(36(a,1), 37(a,1,1,2)), \text{rewrite}([36(2)])\].
142 \[x v (c(x) v y) v z = x v z. \] \[\text{para}(8(a,1), 125(a,1))\].
144 \[C(x, c(x) v y) v c(c(x) v z) v x = x. \] \[\text{para}(58(a,1), 45(a,2,2)), \text{rewrite}([8(26), 93(27)])\].
146 \[c(c_2) v c(c_1) v c(c(c_1) v c(c_2)) = c_1. \] \[\text{hyper}(31,a,34,a), \text{rewrite}([8(9)])\].
148 \[C(x, c(x) v y) v z = x v z. \] \[\text{para}(58(a,1), 37(a,1,2)), \text{rewrite}([8(26), 93(27)])\].
3.2 Proof of 'yCx → xCy' in an orthomodular lattice

Substitute x for y and y for x in Figure 3.1.1.

3.3 Proof of '(xCy ↔ yCx) → Orthomodular Law (OMA)' in an ortholattice
3 C(y,x) <-> C(x,y) # label("Commutativity is symmetric") # label(non_clause). [assumption].
4 y v (c(y) ^ (x v y)) = x v y # label("Symmetry of commutativity implies OMA") # label(non_clause) # label(goal). [goal].
7 x = c(c(x)) # label("AxiL1"). [assumption].
8 c(c(x)) = x. [copy(7), flip(a)].
9 x v y = y v x # label("AxiL2"). [assumption].
12 x v (x ^ y) = x # label("AxiL5"). [assumption].
13 x ^ (x v y) = x # label("AxiL6"). [assumption].
14 c(x) ^ x = 0 # label("AxiOL1"). [assumption].
15 x v c(x) = 1. [copy(14), flip(a)].
16 x v c(x) = 1. [copy(15), rewrite([9(2)])].
30 -C(x,y) | C(c(x) v c(y)) != x # label("Df: commutes"). [clausify(2)].
31 C(x,y) | (x ^ y) v (x ^ c(y)) != x # label("Df: commutes"). [clausify(2)].
32 C(x,y) | c(c(x) v y) v c(c(x) v c(y)) != x. [copy(31), rewrite([17(2), 17(7), 8(4), 9(9)])].
33 C(x,y) | c(c(x) v c(y)) != x # label("Commutativity is symmetric"). [clausify(3)].
34 c1 v c(c1 v c2) != c1 v c2 # answer("Symmetry of commutativity implies OMA"). [deny(4)].
35 c1 v c(c1 v c2) != c1 v c2 # answer("Symmetry of commutativity implies OMA"). [copy(34), rewrite([9(6), 17(7), 8(4), 9(12)])].
36 c(1) = 0. [back_rewrite([14], rewrite([17(2), 8(2), 16(2)])].
37 c(c(x) v (x v y)) = c(x) v y. [back_rewrite([13], rewrite([17(2)])].
38 x v c(c(x) v c(y)) = x. [back_rewrite([12], rewrite([17(2)])].
40 x v (y v z) = y v (x v z). [para([9(a,1), 10(a,1,1,1)], rewrite([10(2)])].
42 x v (c(x) v y) = 1 v y. [para([16(a,1), 10(a,1,1,1)], flip(a)].
53 C(x,1) | c(0 v (c(x) v c(1 v c(x)))) != x. [para([36(a,1), 32(b,1,2,1,2)], rewrite([9(5), 9(9), 9(11)])].
54 c(x) v (c(x) v y) = c(x). [para([37(a,1), 8(a,1,1,1)], flip(a)].
58 c(0 v c(x)) = x. [para([16(a,1), 37(a,1,1,1,1)], rewrite([36(3), 9(3)])].
60 c(x v x v y) | c(1 v y) v x != x. [para([37(a,1), 32(b,1,2,2)], rewrite([40(5), 42(5)])].
61 1 v x = 1. [para([36(a,1), 37(a,1,1,1)], rewrite([58(6)])].
63 C(x,1) | x v 0 != x. [back_rewrite([53], rewrite([58(6), 61(5), 36(4)])].
65 C(x,v x y) | 0 v x != x. [back_rewrite([60], rewrite([61(4), 36(4)])].
74 x v 0 = x. [back_rewrite([16(a,1), 38(a,1,2,1,2)], rewrite([36(2)])].
76 C(x,1). [back_rewrite([63], rewrite([74(4)])].
82 0 v x = x. [hyper([30(a,1), 78(a)], rewrite([9(3), 61(3), 36(2), 36(4), 9(4), 58(5)])].
85 C(x, v x y). [back_rewrite([65], rewrite([92(4)])].
107 C(x v y, x). [hyper([33(a,85)], a)].
111 x v (c(x v y) v x v y). [hyper([30(a,1), 107(a)], rewrite([9(3), 54(8), 8(6), 9(5)])].
112 $F # answer("Symmetry of commutativity implies OMA"). [resolve([111(a,35)], a)].

------------------------------- end of proof -------------------------------

Figure 3.3.1. Summary of a prover9 ([2]) proof of the proposition 'xCy <-> yCx') -> OMA in an ortholattice.

4.0 Conclusions and discussion

The results in Section 3 motivate several observations:

1. The combination of the proofs in Sections 3.1 and 3.2 constitutes a proof of the SoCT in an orthomorphic lattice.

2. The proof in Section 3.3 shows that symmetry of commutativity in an ortholattice implies the OMA.

3. Sections 3.1, 3.2, and 3.3 collectively show that the axioms of an ortholattice, together with the SoCT, implies the OMA, and the axioms of an
orthomodular lattice imply the SoCT. This result is equivalent to one of the two principal propositions of the Foulis-Holland Theorem ([7], [8]). In this sense, the SoCT is equivalent to the OMA.

4. The proofs in Section 3 appear to be novel.

5.0 Acknowledgements

This work benefited from discussions with Tom Oberdan, Frank Pechioni, Tony Pawlicki, and the late John K. Prentice, whose passion for foundations of physics inspired those of us privileged to have known him. For any infelicities that remain, I am solely responsible.

6.0 References