Topological Mixing Derived From Glider $D_1$ of Universal ECA Rule

Fang Wang, Fangyue Chen, Lingxiao Si, and Pingping Liu
School of Science, Hangzhou Dianzi University, Hangzhou, Zhejiang, P. R. China

Abstract—Rule 110 is a complex cellular automaton (CA) in Wolfram’s system of identification and capable of supporting universal computation. There is no doubt that the dynamical property of rule 110 is extremely complex and still not well understood. Gliders in one-dimensional cellular automata are compact groups of non-quiescent and non-ether patterns translating along automaton lattice. This paper reveals a interesting relation between the complexity and the glider in rule 110, and rigorously proves that rule 110 is topological mixing and is chaotic in the sense of both Li-Yorke and Devaney on the subsystem derived from the existing glider $D_1$.

Keywords: cellular automata; glider; topologically mixing; chaos; directed graph.

1. Introduction

Cellular automata (CA) was introduced by von Neumann and Ulam in the late 1940s and early 1950s [1]. In late 1960s, Conway proposed his now-famous Game of Life, which shows the great potential of CA in the simulation of complex systems [2]. Mathematical theory of CA was developed by Hedlund about two decades after Neumann’s work. He studied CA in the context of symbolic dynamics and the sequences in $S^*$ denote a set of some finite sequences over $S$. In [6], he classified the 256 elementary cellular automata (ECA) informally into four classes using dynamical concepts like periodicity, stability and chaos.

Based on Wolfram’s work, Chua et al. provided a rigorous nonlinear dynamical approach to his empirical observations based on mathematical analysis [7-10]. Although there are 256 elementary cellular automata (ECA) rules, only 88 rules are globally independent from each other [7-8, 15]. These 88 global independent ECA rules are also organized into 4 groups with distinct qualitative dynamics: 40 period-$k$ ($k = 1, 2, 3, 6$), 30 topologically distinct Bernoulli shift rules, 10 complex Bernoulli shift rules and 8 hyper Bernoulli shift rules [7-8].

Among infinitely many local CA rules, rule 110 in Wolfram’s system of identification has received special attention. One of the first investigations about rule 110 was described by Wolfram, discovering that the rule displays complex behaviors by means of the existence of gliders—a glider is a periodic structure moving into the evolution space from random initial conditions. Rule 110, like the Game of Life, is named as left life by Cook. He showed new gliders with rare extensions and a pair of gliders of complicated constructions [11]. Also, he demonstrated that rule 110 was universal via simulating a cyclic tag system with well-defined blocks of gliders by means of collisions [12, 20, 21].

This paper mainly focuses the complexity of rule 110 on a subsystem which is derived from the existing glider $D_1$. The rest of the paper is organized as follows: Section 2 reviews the basic concepts of one-dimensional CA and symbolic dynamics. Section 3 identifies a subsystem of rule 110 which is a subshift of finite type, and discusses its complex dynamics. Section 4 concludes the present work.

2. Preliminaries

A bi-infinite sequence, $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$, is called a configuration, where $x_i \in S = \{0, 1, \cdots, k-1\}$ and the star “$*$” denotes the designated symbol of $x$. If $I = [i, j]$ is an interval integers, put $x_{[i,j]} = (x_i, x_{i+1}, \cdots, x_j)$ ($i < j$), $x_{[i,j]} = (x_i, \cdots, x_{j-1})$. For a finite sequence $a = (a_0, a_1, \cdots, a_{n-1})$, if there exists an $m \in Z$ such that $x_{m+k} = a_k$ ($k = 0, 1, \cdots, n-1$), then it is said that $a$ appears in $x$, denoted as $a \prec x$. The set of configurations is denoted by $S^Z$ and a metric “$d$” on $S^Z$ is defined as

$$d(x, y) = \sup \left\{ \frac{\rho(x_i, y_i)}{2^n} \mid i \in Z \right\}, \rho(x_i, y_i) = \begin{cases} 0, & x_i = y_i, \\ 1, & x_i \neq y_i, \end{cases}$$

where $\rho(\cdot, \cdot)$ is the metric on $S$ and $x, y \in S^Z$. It is known that $(S^Z, d)$ is a compact, perfect, and totally disconnected metric space [13].

Let $f$ denote a self map on $S^Z$. A subset $X \subseteq S^Z$ is $f$-invariant if $f(X) \subseteq X$, and strongly $f$-invariant if $f(X) = X$. If $X$ is closed and $f$-invariant, then $(X, f)$ or simply $X$ is called a subsystem of $(S^Z, f)$.

The left-shift map $\sigma_L : S^Z \to S^Z$ is the homeomorphism defined by $[\sigma_L(x)]_i = x_{i+1}, \forall x \in S^Z, i \in Z$, where $[\sigma_L(x)]_i$ stands for the $i$-th element of $\sigma_L(x)$. The restriction of $\sigma_L$ to any closed and strongly $\sigma_L$-invariant subset $\Lambda \subseteq S^Z$ is called a subshift, denoted by $(\Lambda, \sigma_L)$. For instance, let $\sigma'$ denote a set of some finite sequences over $S$, and $\Lambda = \Lambda_{\sigma'}$ is the set of configurations made consisting of the sequences in $\sigma'$. Then $\Lambda_{\sigma'}$ is a subsystem of $(S^Z, \sigma_L)$, where $\sigma'$ is said to be the determinative block system of $\Lambda$. Meanwhile, the right-shift map $[\sigma_R] : S^Z \to S^Z$ defined by $[\sigma_R(x)]_i = x_{i-1}, \forall x \in S^Z, i \in Z$, has similar results as $\sigma_L$.

A map $f : S^Z \to S^Z$ is a CA if it is continuous and commutes with $\sigma$, where $\sigma$ is left-shift or right-shift map.
For any CA, there exist a radius \( r > 0 \) and a local map \( f : S^{2r+1} \to S \) such that \( [f(x)]_i = f(x_{i-r,r+i}) \). \( f \) is an ECA when \( r = 1 \).

Each ECA can be expressed by a 3-bit Boolean function and coded by an integer \( N \), which is the decimal notation of the output binary sequence of the Boolean function [7]. The local expression of rule 110 is

\[
\begin{align*}
\hat{f}(0, 0, 0) &= 0 & \hat{f}(1, 0, 0) &= 0 \\
\hat{f}(0, 0, 1) &= 1 & \hat{f}(1, 0, 1) &= 1 \\
\hat{f}(0, 1, 0) &= 1 & \hat{f}(1, 1, 0) &= 1 \\
\hat{f}(0, 1, 1) &= 1 & \hat{f}(1, 1, 1) &= 0
\end{align*}
\]

i.e., the evolution rule is expressed in binary notation as \( 01110110 \) (representing the decimal number \( N = 110 \), where the leftmost one is the low position and the rightmost one is the high position). The global map of rule 110 is denoted by \( f_{110} \). The evolution of the automaton is defined starting from linear array of cells each containing one state of \( S = \{0, 1\} \); taking every cell \( x_i \) as a central one, to evaluate the value of its corresponding neighborhood so as to determine the new central element in the following generation:

\[ \hat{f}(x_{i-1}, x_i, x_{i+1}) \rightarrow x_{i+1} \]

Time \( t \) is discrete and there is a simultaneous evaluation of each \( x_i \) in the array, i.e., parallel mappings generate the next array, determining the evolution space \( S^Z \).

3. Glider and Subsystem

3.1 Glider \( D_1 \)

A glider is a compact group of non-quiescent states traveling along CA lattice, and is a periodic structure moving in time [6, 11-12, 20-21].

It is known that the speed of glider \( D_1 \) of rule 110 is \( 2/10 \), the lineal volume is \( 11-25 \), and the even and odd number of periodic margin on the left (or right) border in the ether pattern are 1 and 2 respectively [21]. Under the viewpoint of symbolic dynamics, the ether pattern and glider can be defined as the evolutionary orbit starting from a special initial configuration [22]. The ether factor of the ether patterns is \( a = (1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0) \), and one of glider factors of \( D_1 \) is \( b = (1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0) \), i.e., an ether pattern of Rule 110 is the evolutionary orbit \( \text{Orb}_{f_{110}}(a^*) = \{a^*, f_{110}(a^*), f_{110}^2(a^*), \ldots \} \) and glider \( D_1 \) is the evolutionary orbit \( \text{Orb}_{f_{110}}(\bar{x}) = \{\bar{x}, f_{110}(\bar{x}), f_{110}^2(\bar{x}), \ldots \} \) in the CA lattice space, where \( a^* = (\cdots, a, a, a, \cdots) \) is a cyclic configuration and \( \bar{x} = (\cdots, a, a, b, a, a, \cdots) \). That the speed of \( D_1 \) is \( 2/10 \) implies this glider shifts to right by 2 bits in every 10 iterations under rule 110, i.e., \( f_{110}^{10}(\bar{x}) = \sigma_{R}^2(\bar{x}) \). The ether pattern and glider \( D_1 \) are shown in Figure 1.

![Fig. 1: Ether pattern and glider D1 of rule 110.](image)

3.2 Subsystem

One can obtain a 21-sequence set \( \mathcal{A}_1 = \{q \mid q = x_{[i, i+20]}, \forall i \in Z \} \) from \( \bar{x} = (\cdots, a, a, a, b, a, a, \cdots) \), where \( a = (1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0) \) is the ether factor and \( b = (1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0) \) is a glider factor of \( D_1 \). More specifically \( \mathcal{A}_1 = \{00000010110110110101, 00100010110110110101, 00100101110110110101, 00100110110110110101, 00100111110110110101, 00110110111011100101, 00111100001011011011, 00111100010110110111, 00111101001111110001, 100011011111110001, 100110111111111101, 10011111100010111111, 10011111110001111111, 10011111111001111111, 10011111111100011111, 10011111111110011111, 10011111111111001111, 10011111111111101111, 10011111111111111111, 10011111111111111111\}

For simplicity, \( \mathcal{A}_1 \) is expressed by its decimal code \( D(\mathcal{A}_1) = \{159469, 159625, 318939, 319251, 454367, 456859, 637879, 638502, 908734, 913719, 974611, 978417, 1018299, 1018302, 1128310, 1128388, 1275759, 1277005, 1505435, 1535881, 1537784, 1557725, 1557727, 1612731, 1612770, 1801293, 1816516, 1817468, 1827439, 1827439, 1854941, 1854961, 1949222, 1956834, 1976046, 1976056, 2036599, 2036604\}.

**Proposition 1:**

(1) For rule 110, there exists a subset \( \Lambda_1 \subset S^Z \), such that \( f_{110}^{10}(x) = \sigma_R^2(x) \) for \( x \in \Lambda_1 \), where \( \Lambda_1 = \Lambda_{\mathcal{A}_1} = \{x \in S^Z \mid x_{[i-10, i+10]} \in \mathcal{A}_1, \forall i \in Z \} \), and \( \mathcal{A}_1 \) is its determinative block system.

(2) Let

\[ \Lambda = \bigcup_{i=0}^{9} f_{110}^i(\Lambda_1), \]

then, \( \Lambda \) is a \( f_{110}^{10} \)-invariant subset of \( S^Z \), and \( f_{110}^{10}(x) = \sigma_R^2(x) \) for \( x \in \Lambda \).

(3) \( \sigma_R : \Lambda \to \Lambda \) is a subshift of finite type (SFT).
Let $\Lambda = \Lambda_{\mathcal{A}}$, $\mathcal{A}$ is the determinative block system of $\Lambda$, then, $\mathcal{A}$ is a 21-sequence set consisting of 237 elements. Due to space limitations and for simplicity, the decimal code set $D(\mathcal{A})$ of $\mathcal{A}$ is placed in Appendix I.

### 3.3 Finite directed graph and complexity

Since $(\Lambda_{\mathcal{A}}, \sigma_R)$ is a SFT, then $\Lambda_{\mathcal{A}}$ can be described by a finite directed graph $G_{\mathcal{A}} = (\mathcal{A}, \mathcal{E})$, in which each vertex is labeled by a sequence in $\mathcal{A}$, and $\mathcal{E}$ is the edge set. Two vertices $a = (a_0, a_1, \cdots, a_{n-1})$ and $b = (b_0, b_1, \cdots, b_{n-1})$ are connected by an edge of $\mathcal{E}$ if and only if $a_k = b_{k-1}, k = 1, 2, \cdots, n-1$. Every edge $(a_0, a_1, \cdots, a_{n-1}) \rightarrow (b_0, b_1, \cdots, b_{n-1})$ of $\mathcal{E}$ is labeled by $b_{n-1}$. One can think of each element of $\Lambda_{\mathcal{A}}$ as a bi-infinite path on the graph $G_{\mathcal{A}}$. Whereas a directed graph corresponds to a square transition matrix $A = (A_{ij})_{m \times n}$ with $A_{ij} = 1$ if and only if there is an edge from vertex $b^{(i)}$ to vertex $b^{(j)}$, where $m = |\mathcal{A}|$ is the number of elements in $\mathcal{A}$, and $i$ (or $j$) is the code of the vertex in $\mathcal{A}$, $i, j = 0, 1, \cdots, m-1$. Thus, $\Lambda_{\mathcal{A}}$ is precisely defined by the transition matrix $A$. Transition matrix $A = (A_{ij})_{237 \times 237}$ is shown in Appendix II.

A square matrix $A$ is irreducible if, for any $i, j$, there exists an $n$ such that $A^n_{ij} > 0$; aperiodic if there exists an $n$, such that $A^n_{ij} > 0$, for all $i, j$. Where $A^n_{ij}$ is the $(i, j)$ entry of $A^n$ [13, 14].

**Lemma 1:**

1. If $\Lambda_{\mathcal{A}}$ is a SFT, then $\sigma$ ($\sigma_R$ or $\sigma_L$) is topologically transitive if and only if $A$ is irreducible; topologically mixing if and only if $A$ is aperiodic [13].
2. $A$ is irreducible if and only if for every ordered pair of vertices $b^{(i)}$ and $b^{(j)}$ in $\mathcal{A}$ there is a path in the graph $G_{\mathcal{A}}$ starting at $b^{(i)}$ and ending at $b^{(j)}$; $A$ is aperiodic if and only if it is irreducible and the numbers of the length of two different closed paths in the graph $G_{\mathcal{A}}$ are coprime [13-14, 16].

**Proposition 2:**

1. $\sigma_R$ is topologically mixing on $\Lambda$;
2. $\sigma_R^2 = f_{110}^2$ is topologically mixing on $\Lambda$;
3. $f_{110}$ is topologically mixing on $\Lambda$;
4. The topological entropy of $f_{110}|\Lambda$ is positive.

**Proof:**

1. By computer-aided methods, it is easily found that there exist two different closed paths in the graph $G_{\mathcal{A}} = (\mathcal{A}, \mathcal{E})$, and the numbers of their length are coprime (one is 14 and another is 25, as shown in Fig. 2), thus, the transition matrix $A$ corresponding to the graph is aperiodic, so the shift $\sigma_R$ is mixing;
2. The conclusion is obvious;
3. One only need to prove that for any nonempty open subsets $U, V \subset \Lambda$, $\exists N_0 > 0$, $f_{110}^n(U) \cap V \neq \emptyset$ for any $n \geq N_0$.

Fig. 2: Two different closed paths in the $G_{\mathcal{A}} = (\mathcal{A}, \mathcal{E})$

It is well known that positive topological entropy implies chaos in the sense of both Li-Yorke [13-14, 16-19], and topologically mixing property implies many chaotic properties in different senses such as Devaney [16, 18]. Thus, one has the interesting result.

**Theorem 1:** $f_{110}$ is chaotic in the sense of both Li-Yorke and Devaney on $\Lambda$.

### 4. Conclusion

In this paper, some topological dynamics of rule 110 has been discussed under the framework of symbolic dynamical systems. It is proved that rule 110 is topological mixing and possesses positive topological entropy and is chaotic in the sense of both Li-Yorke and Devaney on a subsystem derived from a existing glider of rule 110. Nevertheless, the complete symbolic dynamical properties of rule 110 are still an open problem, some new methods should be exploited to investigate its dynamical behaviors in future studies.

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References


Appendix I

The determinative block system $\mathcal{A}$ of $\Lambda$ is a 21-sequence set consisting of 237 elements. For a sequence $a = (a_0, a_1, \ldots, a_{20}) \in \mathcal{A}$, its decimal code is defined as

$$D(a) = \sum_{i=0}^{20} a_i \cdot 2^{20-i}$$

For simplicity, $\mathcal{A}$ is replaced by its decimal code set $D(\mathcal{A}) =$

(79812, 97357, 146509, 157261, 159373, 159469, 159621, 159624, 159625, 159630, 159673, 159675, 159693, 194715, 228429, 240717, 243789, 247007, 293019, 314523, 318747, 319243, 319249, 319251, 319347, 319351, 319387, 389431, 413169, 419039, 452831, 454367, 456799, 456847, 456875, 456859, 456939, 456942, 457631, 457663, 457951, 481435, 487579, 494014, 509151, 581395, 586039, 629047, 637495, 637879, 638487, 638499, 638502, 638522, 638523, 638695, 638703, 638775, 725884, 778863, 806385, 826338, 838078, 905662, 908734, 913598, 913694, 913715, 913719, 913878, 913884, 915262, 915326, 915902, 922492, 949325, 953841, 962871, 974611, 975159, 978417, 982093, 988028, 1017329, 1018097, 1018265, 1018298, 1018299, 1018302, 1018303, 1019569, 1019617, 1030641, 1031153, 1033439, 1035761, 1088482, 1097254, 1121830, 1127206, 1128262, 1128310, 1128386, 1128388, 1128391, 1128412, 1128413, 1128422, 1128424, 1162790, 1168934, 1170470, 1172079, 1208201, 1255160, 1258095, 1274991, 1275759, 1276975, 1276999, 1277004, 1277005, 1277045, 1277047, 1277391, 1277407, 1277551, 1303151, 1339273, 1411518, 1451768, 1505435, 1509822, 1522328, 1525496, 1535881, 1537784, 1539622, 1557240, 1557624, 1557708, 1557725, 1557727, 1558360, 1558384, 1563896, 1564152, 1565295, 1566456.
Appendix II

The index $(i, j)$ set of the transition matrix $A = (A_{ij})_{237 \times 237}$ with $A_{ij} = 1, 1 \leq i, j \leq 237$ corresponding to the determinative block system $\mathcal{A}$ is as follows.

\{(1,8), (1,9), (2,14), (3,19), (4,20), (5,21), (6,22), (7,23), (8,24),
(9,25), (10,26), (11,27), (12,28), (13,29), (14,30), (15,38), (16,44),
(17,45), (18,46), (19,49), (20,50), (21,51), (22,52), (23,53),
(24,54), (25,55), (26,56), (26,57), (27,58), (28,59), (29,60), (30,62),
(31,64), (32,65), (33,66), (34,67), (35,68), (36,69), (37,70), (38,71),
(39,72), (40,73), (41,74), (42,75), (43,76), (44,80), (45,82), (46,85),
(47,91), (47,92), (48,111), (49,114), (50,117), (51,118), (52,119),
(53,120), (54,121), (55,122), (55,123), (56,124), (57,125), (58,126),
(59,127), (60,128), (61,132), (62,144), (63,158), (63,159), (64,164),
(65,165), (66,173), (67,175), (68,177), (69,178), (70,179), (71,180),
(71,181), (72,182), (73,183), (74,184), (75,185), (76,187), (77,188),
(78,201), (79,202), (80,203), (81,204), (82,205), (83,207), (84,209),
(85,215), (86,223), (87,224), (88,225), (89,226), (90,227), (91,228),
(91,229), (92,230), (93,231), (94,232), (95,234), (96,235), (97,236),
(98,237), (99,1), (100,2), (101,3), (102,4), (103,5), (104,6), (105,7),
(106,8), (106,9), (107,10), (108,11), (109,12), (110,13), (111,15),
(112,16), (113,17), (114,18), (115,25), (116,31), (117,32), (118,33),
(119,34), (120,35), (121,36), (122,37), (123,38), (124,39), (125,40),
(126,41), (127,42), (128,43), (129,47), (130,48), (131,61), (132,63),
(133,71), (134,77), (135,78), (136,79), (137,81), (138,83), (139,84),
(140,86), (141,87), (142,88), (143,89), (143,90), (144,91), (144,92),
(145,93), (146,94), (147,95), (148,96), (149,97), (150,98), (151,99),
(152,100), (153,101), (154,102), (155,103), (156,104), (157,105),
(158,106), (159,107), (160,108), (160,109), (161,110), (162,112),
(163,113), (164,115), (165,116), (166,122), (166,123), (167,129),
(168,130), (169,131), (170,133), (171,134), (172,135), (173,136),
(174,137), (175,138), (176,139), (177,140), (178,141), (179,142),
(180,143), (181,144), (182,145), (183,146), (184,147), (185,148),
(186,149), (187,150), (188,151), (189,152), (190,153), (191,154),
(192,155), (193,156), (194,157), (195,158), (195,159), (196,160),
(197,161), (198,162), (199,163), (200,166), (201,167), (202,168),
(203,169), (204,170), (205,171), (206,172), (207,174), (208,176),
(209,186), (210,189), (211,190), (212,191), (213,192), (214,193),
(215,194), (215,195), (216,196), (217,197), (218,198), (219,199),
(220,200), (221,206), (222,208), (223,210), (224,211), (225,212),
(226,213), (227,214), (228,215), (229,216), (230,217), (231,218),
(232,219), (233,220), (234,221), (235,222), (236,228), (236,229),
(237,233)).