A Vertical Splitting Scheme for Nonhydrostatic Atmospheric Model

A. Bourchtein, and L. Bourchtein
Institute of Physics and Mathematics, Pelotas State University, Pelotas, Brazil

Abstract - In this study, a numerical solution of nonhydrostatic atmospheric equations is considered. A robust semi-implicit approach with additional time splitting is applied in order to construct computationally efficient and accurate numerical scheme for modeling of large-scale atmospheric dynamics. Description of the designed numerical algorithm is provided and its accuracy and stability are discussed. The main properties of the scheme are compared to the respective properties of more traditional algorithms. Performed numerical experiments with the actual atmospheric data of pressure, temperature and wind show that the developed scheme supplies accurate forecast fields for the increased time steps chosen in accordance with the physical requirements.

Keywords: atmospheric modeling, numerical solution, semi-implicit scheme, time splitting

1 Introduction

The Earth atmosphere is a complex dynamical system which supports the processes of different time and space scales, including the most important synoptic (weather) processes. There are different approximated physical models of atmosphere dynamics constructed to filter the secondary waves, however all these approximations introduce certain distortions to the principal synoptic modes. For this reason and also due to considerable increasing of computer power and advances in numerical methods, in the last years a great attention in atmospheric modeling is given to non-filtered models called also nonhydrostatic equations. These models include the Euler momentum equations, mass conservation equation and energy conservation equation for compressible inviscid ideal gas considered usually in the rotating frame related to the Earth surface. Analysis of the corresponding linearized nonhydrostatic equations reveals three kinds of the waves - acoustic, gravity and inertial waves - from which only the last waves are related to the synoptic processes. These waves differ not only in the source of their origin (compressibility, gravity force and Coriolis force together with advection), but also in such important characteristics as propagation velocity and energy contribution. In fact, the characteristic propagation speed of the acoustic waves in the Earth's atmosphere is about 330-340 m/s, while the speed of the gravity waves is always below 330 m/s (with majority of the gravity waves propagating at 100 m/s and below) and the propagation speed of the inertial processes (advection and Rossby waves) is usually about 10 m/s with the (very rare) maximum values below 80 m/s [6,9,11]. At the same time, it is well-known from evaluations of the spectrum of oscillations of the Earth atmosphere that the energy of the acoustic waves is negligible, the gravity modes contain a small part of the evaluable energy for the majority of the large- and meso-scale processes, and the inertial modes contain the main part of the atmospheric energy [9,11]. In this way, the problem of numerical weather forecasting and atmospheric modeling is a stiff problem.

Since the wave filtration can not be performed in the differential form without distortion of the principal inertial modes, the problem stiffness should be addressed in the design of numerical algorithm in order to achieve sufficient efficiency and accuracy of numerical solution. Indeed, the time step of a numerical scheme is generally determined by the Courant-Friedrichs-Lewy (CFL) condition [7]:

\[ \tau \leq \frac{h}{c_{\text{max \, exp}}} \]

where \( \tau \) is the time step, \( h \) is the mesh size of a spatial grid and \( c_{\text{max \, exp}} \) is the maximum propagation speed of the processes treated explicitly in the chosen form of the time differencing. For many totally implicit schemes there is no restriction on the size of the time step due to
makes the fully explicit time differencing equal to the propagation speed defined by the CFL condition with the maximum speed at atmospheric models is too small. For such schemes it is Implicit differencing of all the main linear terms in integration, but the size of the time step for non-filtered and fast calculations at each time step of numerical systems at each time step, that turns these schemes excessively computationally demanding and inefficient. For this reason, totally implicit time differencing is not practically used in atmospheric modeling.

On the other pole, fully explicit schemes assure simple and fast calculations at each time step of numerical integration, but the size of the time step for non-filtered atmospheric models is too small. For such schemes it is defined by the CFL condition with the maximum speed equal to the propagation speed $c_{max,acoust}$ of the acoustic waves [7,15]. Therefore, for the typical vertical resolution $h_v = 400 \text{ m}$ used in large- and meso-scale models, the CFL condition

$$\tau \leq \frac{h_v}{c_{max,acoust}}$$

requires the time steps less than 1 sec. Taking in account that the characteristic time scale of the synoptic (large- and meso-scale) models is about a couple of hours, such severe restriction on $\tau$ makes the fully explicit time differencing computationally inefficient.

A more reasonable approach is a semi-implicit time differencing, in which the linear terms of the governing equations, which are responsible for the fastest waves, are approximated implicitly, while the remaining terms - explicitly. For example, applying an implicit approximation to the main linear terms in the vertical, it is possible to increase the time step almost two orders, because the CFL condition will include the horizontal propagation speed of the gravity waves $c_{max,grav}$ [13, 15]. For the typical horizontal resolution $h_h = 20 \text{ km}$, the time restriction

$$\tau \leq \frac{h_h}{c_{max,grav}}$$

will allow the time steps up to 1 min. The price of this improvement is a necessity to solve systems of linear equations with narrow-band matrices at each time step, which can be accomplished efficiently employing the Gelfand-Thomas type algorithm [7]. However, the above time step is still too small as compared to the accuracy requirements.

Implicit differencing of all the main linear terms in the governing equations allows further increasing of the time step up to 5 min (under the same horizontal resolution $h_h = 20 \text{ km}$) [7,14]: because only the maximum advection speed $c_{max,adv}$ enters in the CFL condition in this case. This is a reasonable choice for the time step, comparable with the physical requirements of accuracy. However, such time approximation leads (at each time step) to solution of high-order linear systems with wide-band matrices, that pull down the efficiency of this approach.

It is worth to note that the semi-implicit approach assures usually the same level of accuracy of numerical solutions as the explicit schemes with the same approximation order. This is the reason why the semi-implicit schemes are so popular in atmospheric modeling. Nevertheless, when these schemes arrive to the limit of their efficiency, some additional techniques should be applied to speed-up computations. One of such techniques is a splitting by physical processes. Applied in its extreme form, it usually causes substantial additional errors, which grows significantly when the time step approaches the limit allowable by the CFL condition. In such cases a numerical splitting scheme can be stable for rather large time steps and keeps formally the required approximation order, but the practical accuracy of numerical solutions can fall down [7, 13, 15]. Therefore, the splitting techniques should be applied cautiously, frequently using a partial splitting, and checking the accuracy of the obtained results against available atmospheric data and simulation output of other verified models.

In this study, a partial physical splitting is introduced in the semi-implicit scheme in such a way that all the vertical modes are separated according their propagation velocity. Such vertical splitting corresponds to the analytical separation of the spectrum of the atmospheric waves supported by the governing equations. The gravity waves of the fast vertical modes are approximated implicitly, while those of the slow vertical modes are treated explicitly. It allows us to substitute the problem of solution of three-dimensional elliptic equations by solution of a set of the two-dimensional elliptic problems. The solution of the last set of problems can be obtained much more efficiently than for their three-dimensional counterpart. Additionally, a simpler approximation can be applied to slower vertical modes, because they contain a small part of the total energy [9, 16]. The performed numerical experiments show that applied vertical splitting maintain the desired level of the accuracy of the forecasting fields and speeds-up computations required for each forecast.
2 Governing nonhydrostatic equations

The momentum equations of inviscid atmosphere in the coordinate system related to the rotating Earth can be written as follows:

\[ u_t = \tilde{f}v - R\tilde{T}p_x + N_u, \]
\[ v_t = -\tilde{f}u - R\tilde{T}p_y + N_v, \]  
\[ w_t = \frac{g}{\tilde{T}}T - R\tilde{T}p_z + N_w. \]  

(1)

The continuity equations for compressible ideal gas is

\[ P_t = -\frac{c_p}{c_v}(u_x + v_y + w_z) + \frac{\rho}{RT} w + N_p. \]  

(3)

The last equation is the thermodynamic one, including the equations of the state for ideal gas,

\[ T_t = \frac{R\tilde{T}}{c_p}P_t - \frac{\rho}{c_p} w + N_T. \]  

(4)

The following notations are used above:

- the independent variables: \( t \) - the time coordinate, \( x, y, z \) - the spatial Cartesian coordinates;
- the unknown functions: \( u, v, w \) - the velocity components, \( P = \ln p \) - the pressure logarithm, \( T \) - the temperature;
- the given parameters: \( f \) - the Coriolis parameter with the mean value \( \tilde{f} = const \), \( g \) - the gravitational acceleration, \( \tilde{T} = const \) - the mean value of the temperature, \( R \) - the gas constant, \( c_p \) and \( c_v \) - the specific heat at constant pressure and volume, respectively;
- the nonlinear terms \( N_u, N_v, N_w, N_T \) represent mainly advective terms in each equation, whose specific form will not be used in the subsequent formulas.

This is the standard form of the governing equations of the nonhydrostatic atmosphere, which can be found in different sources (e.g. [7,11]).

3 Time splitting semi-implicit scheme

In this section the time splitting semi-implicit scheme is presented and its analytical properties of accuracy and stability are discussed and compared to the respective properties of the standard semi-implicit scheme.

3.1 Standard semi-implicit scheme

The standard three-time-level semi-implicit time differencing of the second order of accuracy for equations (1)-(4) has the following form [8, 14, 15]:

\[ \frac{u^\tau - u^{-\tau}}{2\tau} = \tilde{f} \frac{v^\tau + v^{-\tau}}{2} - \frac{RT}{2} \frac{P^{\tau} + P^{-\tau}}{2} + N_u, \]
\[ \frac{v^\tau - v^{-\tau}}{2\tau} = -\tilde{f} \frac{u^\tau + u^{-\tau}}{2} - \frac{RT}{2} \frac{P^{\tau} + P^{-\tau}}{2} + N_v, \]
\[ \frac{w^\tau - w^{-\tau}}{2\tau} = \frac{g}{\tilde{T}} \frac{T^{\tau} + T^{-\tau}}{2} - \frac{RT}{2} \frac{P^{\tau} + P^{-\tau}}{2} + N_w, \]
\[ \frac{P^{\tau} - P^{-\tau}}{2\tau} = -\frac{\rho}{c_v} \frac{D^\tau + D^{-\tau}}{2} c_p \frac{w^{\tau} + w^{-\tau}}{2} + \frac{\rho}{c_p} \frac{w^{\tau} + w^{-\tau}}{2} + N_p. \]  

(5)

Here \( D = u_x + v_y \) - the horizontal divergence, \( \tau \) - the time step, the superscript "\( \tau \)" denotes the quantities on the next time level \( t_{n+1} = (n+1)\tau \) (so-called the prognostic values), the superscript "\( -\tau \)" denotes the values on the past time level \( t_{n-1} = (n-1)\tau \), and functions without superscripts are considered on the current time level \( t_n = n\tau \):

\[ \varphi^\tau = \varphi(t_{n+1}, x, y, z), \varphi = \varphi(t_n, x, y, z), \varphi^{-\tau} = \varphi(t_{n-1}, x, y, z), \varphi = u, v, w, P, T. \]

It is easy to realize that the scheme (5)-(8) combines the implicit Crank-Nicolson approximation with the double time step for the linear terms, and the explicit leapfrog approximation for the nonlinear and variable coefficient terms.

The analysis of linear stability of the scheme (5)-(8) shows that the time step is restricted by the condition

\[ \tau \leq \frac{h_h}{c_{max\; adv}}. \]

For the horizontal mesh size \( h_h = 20 \) km and the maximum advection velocity \( c_{max\; adv} = 60 \) m/s, the maximum allowable time step is about 5 min.

3.2 Splitting scheme: semi-implicit step

Although the scheme (5)-(8) is rather efficient as compared to more explicit schemes, it has some drawbacks, the main of which is a necessity to perform inefficient computations at each time step related to solution of three-dimensional elliptic problems arising due to the implicit approximation of the linear terms. Since a part of these terms is responsible for slowly propagating internal gravity waves, the amount of computations can be decreased by applying vertical splitting. Such splitting allows us to separate and, consequently, approximate differently the principal (fast) and secondary (slow)
vertical modes. A similar approach has been applied successfully in the hydrostatic atmospheric models [3, 4, 5, 10, 12]. To perform the vertical splitting, it is convenient to divide each time step in two stages. On the first stage, the explicit leap-frog scheme is applied to all the terms, except for those responsible for vertical propagation of the acoustic waves, which are approximated implicitly by the Crank-Nicolson scheme:

\[
\frac{\hat{u}^* - u^-}{2\tau} = \frac{\hat{v} - R\hat{T}P_y + N_u},
\]

\[
\frac{\hat{v}^* - v^-}{2\tau} = -\frac{\hat{u} - R\hat{T}P_y + N_v},
\]

\[
\frac{\hat{w}^* - w^-}{2\tau} = \frac{g}{T} \frac{\hat{T}^* + T^-}{2} - \frac{R\hat{T}_z^* + P_y^*}{2} + N_w .
\]

\[
\frac{\hat{P}^* - P^-}{2\tau} = -\frac{c_p}{c_v} D - \frac{c_p}{c_v} \frac{\hat{w}^* + w^-}{2} + \frac{g}{RT} \frac{\hat{w}^* + w^-}{2} + N_p .
\]

\[
\frac{\hat{T} - T^-}{2\tau} = \frac{RT}{c_p} \frac{\hat{T}^* - P^-}{2\tau} - \frac{g}{c_p} \frac{\hat{w}^* + w^-}{2} + N_T .
\]

The computations on this stage are much faster than for the standard semi-implicit scheme (5)-(8). Indeed, the last three equations can be decoupled from the system and solved separately. Unknown functions in these equations are coupled only in the vertical variable, that reduces (10)-(12) to a set of decoupled one-dimensional boundary value problems, each of which can be solved efficiently by the Gelfand-Thomas algorithm [7]. After solution of (10)-(12) is found, the remaining equations (9) have the explicit form with respect to the horizontal velocity components. The deficiency of the scheme (9)-(12) is its weak stability [8, 13]:

\[
\tau \leq \frac{h_h}{c_{\text{max} \cdot \text{grav}}},
\]

that means that for the horizontal mesh size \(h_h = 20 \text{ km}\) and the maximum propagation speed of the gravity waves \(c_{\text{max} \cdot \text{grav}} = 300 \text{ m/s}\), the maximum allowable time step is about 1 min.

### 3.3 Splitting scheme: correction equations

To improve the stability of (9)-(12), a more implicit time differencing should be applied, for example, such as in (5)-(8). Equations for the differences (corrections) between solutions (prognostic quantities) of (5)-(8) and (9)-(12), can be written as follows:

\[
u^* - \frac{\hat{v}^*}{2\tau} + \frac{\hat{T}^* P^* y}{2\tau} = L_u,
\]

\[
v^* + \frac{\hat{v}^*}{2\tau} + \frac{\hat{T}^* P^* y}{2\tau} = L_v,
\]

\[
w^* - \frac{\hat{w}^*}{2\tau} + \frac{\hat{T}^* P^* z}{2\tau} = L_w,
\]

\[
P^* + \frac{\hat{P}^*}{c_p} + \frac{\hat{T}^* P^*}{c_p} = L_T,
\]

\[
\rho \phi^* - \frac{\rho}{c_p} \phi^* = L_\phi,
\]

where \(\phi = \phi^* - \phi^*\), \(\phi = u, v, w, T, P\) are the corrections to be found and the linear terms \(L_\phi\) do not include the prognostic values of the standard semi-implicit scheme:

\[
L_u = -\frac{\hat{v}^*}{2\tau} - \frac{\hat{T}^* P^* y}{2\tau} - \frac{\hat{T}^* P^* y}{2\tau} + N_u,
\]

\[
L_v = -\frac{\hat{v}^*}{2\tau} - \frac{\hat{T}^* P^* y}{2\tau} + N_v,
\]

\[
L_w = -\frac{\hat{w}^*}{2\tau} - \frac{\hat{T}^* P^* z}{2\tau} + N_w,
\]

\[
L_T = -\frac{\rho}{c_p} \hat{T}^* P^* + \frac{\rho}{c_p} \hat{T}^* P^* + N_T.
\]

Considering, that the prognostic values of the first stage \(\phi^*\) are already calculated, the equations (13)-(16) represent the linear system for corrections \(\phi^*\). Evidently, a solution of this system is equivalent to finding the prognostic values of the standard semi-implicit scheme (5)-(8), but as it was noted above, such procedure requires solution of the three-dimensional elliptic problems. However, application of the vertical splitting can significantly reduce the amount of the computations without loss of accuracy and stability of numerical solution.

The first step to perform the vertical splitting is elimination of unknown functions \(w^*\) and \(T^*\) from equations (14)-(16), that leads to the following equation for \(P^*\):

\[
-\frac{\rho}{c_p} \frac{\hat{T}^* P^* y}{2\tau} + \frac{\hat{T}^* P^* y}{2\tau} + \frac{\rho}{c_p} \frac{\hat{T}^* P^* z}{2\tau} + \frac{\rho}{c_p} \frac{\hat{T}^* P^* z}{2\tau} = L_T .
\]

Considering the first two terms of the last equation together with the kinematic boundary conditions \(w = 0\) on the upper \((z = 0)\) and lower \((z = z_{up})\) boundaries of the atmosphere rewritten for the function \(P^*\), one can arrive to the corresponding Sturm-Liouville problem:

\[
-\frac{\rho}{c_p} \frac{\hat{T}^* P^* y}{2\tau} = \lambda P^*, \quad z \in (0, z_{up}) ;
\]

\[
\frac{\rho}{c_p} \frac{\hat{T}^* P^* y}{2\tau} = 0 , \quad z = 0, \quad z = z_{up} .
\]

It can be shown that the spectrum of the last problem is simple and positive, and the eigenvalues \(c_k\) approach fast the only limit point 0 of the spectrum set. This property is important for a selective correction of the vertical modes.
3.4 Splitting scheme: vertical decoupling

On the second stage of vertical splitting scheme, the eigenfunctions $F_k(z)$ of the problem (18)-(19) are used to expand some unknown and known quantities:

$$\varphi^* = \sum_k \varphi_k^* (x, y) F_k(z), \quad \varphi = u, v, P, L.$$ 

Applying this expansion, the correction equations (13), (17) can be rewritten in the form

$$u^+ - \hat{\varphi} v^+ + \tau \mathbf{R} T \mathbf{P}^+ = L^u_\tau,$$

$$v^+ + \hat{\varphi} u^+ + \tau \mathbf{R} T \mathbf{P}^+ = L^v_\tau,$$

$$\tau \mathbf{R} T \mathbf{P}^+ + \tau c_k^2 \left[ (u^+ + v^+) \right] = L^c_\tau.$$  

(20)

For a simplicity of notations the subscript "k" is omitted in all quantities, except for $c_k$.

Each of the systems (20) can be solved separately from others and represents the time discretization of the linearized shallow water equations with the corresponding gravity wave speed $c_k$. Internal vertical modes, with large values of $k$, correspond to the slow gravity waves and contain a small part of the atmospheric energy. Therefore, they have no influence on the scheme stability and their contribution to the solution accuracy is very small. Hence, it is sufficient to solve only a few of the principal vertical systems (20) in order to improve considerably the stability of the semi-implicit stage (9)-(12). Analysis of the linear stability of the vertically splitting scheme shows that the restriction on the time step can be expressed as

$$\tau \leq \frac{h_k}{c_{I+1}},$$

where $c_{I+1}$ is the maximum propagation velocity of the vertical modes that remain uncorrected. For example, applying corrections to the first four vertical modes in the model with 30 vertical levels, it is possible to increase the time step from 1 min to 5 min, because $c_5 = 50$ m/s. The achieved time step is practically equivalent to the maximum allowable time step of the standard semi-implicit scheme (5)-(8). In this way, in order to recover the scheme stability after the semi-implicit stage (9)-(12), it is sufficient to solve only a small fraction of the systems (20). After this, the inverse vertical transformation returns the physical values of the corrections for the pressure and horizontal components of velocity, and, finally, the corrections to the temperature and vertical velocity are found by explicit formulas (14) and (16).

4 Numerical experiments

The described numerical scheme was applied to forecasting of atmospheric fields on the horizontal area of 3000x3000 km$^2$, covering the South part of Brazil and adjacent territories (the center point was chosen at $30^\circ$ South and $55^\circ$ West) and within the vertical layer of the atmosphere extending from the Earth surface up to 15 km. The horizontal grid was chosen to be uniform with the mesh size of 20 km and the vertical layer was divided in 30 sub-layers of different thickness - the finest vertical resolution was used in the planetary boundary layer and near tropopause, where the vertical variations are the highest.

The initial conditions were derived from the objective analysis fields of the National Centers for Environmental Predictions (NCEP), and the boundary conditions were defined based on the global forecast fields of the same center. The 12, 24 and 36-hour forecasts were calculated on the defined above territory, but the evaluation of the forecast skills was made on the inner territory of the size 1000x1000 km$^2$ in order to minimize the influence of the boundary conditions. The forecasts beyond 36 hours were not calculated, because it is well-known that they are highly dependent on the provided boundary conditions [1,2].

To evaluate the accuracy and efficiency of the constructed scheme, the forecasting results were compared to the respective forecasts obtained with the use of the standard semi-implicit scheme (5)-(8) and the simpler semi-implicit scheme (9)-(12). The standard measures of evaluation of the forecast skills used in the short-range numerical weather prediction were employed: the root-mean-square error (the root-mean-square difference between forecast and analysis fields for the chosen meteorological element and vertical surface) and correlation coefficient between the prognostic and actual tendencies (again for the chosen meteorological element and vertical surface) [1,2].

The root-mean-square errors for the geopotential height of 500 hPa are shown in Fig.1 for the indicated three schemes. This surface is quite characteristic for evaluation of the synoptic processes in mid-troposphere. The root-mean-square errors for the temperature at the pressure surface of 850 hPa are shown in Fig.2. The forecasts on this surface are important for evaluation of convective activity in lower troposphere, which affects the formation of clouds and precipitation in more complete non-adiabatic atmospheric models. One can note a practical identity of the forecast accuracy of the standard and time splitting semi-implicit schemes for the both assessed fields. The evaluations of the correlation coefficients of the forecasts calculated with the indicated
three schemes show similar results (not presented here): the coefficients for the vertically splitting and standard semi-implicit schemes are practically identical, while the coefficient for the scheme (9)-(12) is a bit smaller, showing slightly decreasing quality.

As it can be seen from the provided results, the vertical splitting does not introduce any visible additional errors and provides numerical solutions of the same level of accuracy as the standard semi-implicit scheme: the differences between evaluations of two forecasts are practically negligible for both root-mean-square error and correlation coefficient, and for both chosen meteorological elements. At the same time, the computational time required for the vertical splitting scheme is almost a half of the forecast time for the standard semi-implicit scheme. It is worth to note that according to the properties of the spectrum of the eigenvalue problem (18)-(19), the number of the fast vertical modes remains almost the same when the number of the vertical levels increases. Therefore, under current tendency of enhancing vertical resolution to 40 or 50 vertical levels, or even higher, the computational speed-up obtained in the vertical splitting scheme can be even stronger.

5 Conclusions

In this report we have presented a finite-difference semi-implicit time splitting scheme designed to overcome the problems related to numerical solution of the stiff atmospheric equations based on the non-filtered Euler equations of the ideal compressible gas. Each time step of the constructed scheme consists of two stages: a simple semi-implicit integration and subsequent solution of the corrections equations for the fastest vertical modes. The time step of stable integration is defined by the maximum velocity of processes treated explicitly that allows the use of the time steps comparable with those required by physical accuracy. The provided results of numerical experiments have shown the accuracy of the obtained forecasts and the computational efficiency of the developed numerical scheme.

6 Acknowledgements

This research was supported by Brazilian science foundations CNPq and FAPERGS.

7 References


