Stochastic mixed integer second-order cone programming: A new modeling tool for stochastic mixed integer optimization

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Abstract— *In deterministic mixed integer second-order cone programs (DMISCOPs) we minimize a linear objective function over the intersection of an affine set and a product of second-order (Lorentz) cones, and an additional constraint that requires a subset of the variables attain integers values. We refer to them as* deterministic *mixed integer second-order cone programs since data defining them are deterministic. Stochastic programs have been studied since 1950âĂŹs as a tool to handle optimization problems that involve uncertainty in data. In this paper, we introduce a new modeling tool for stochastic mixed integer optimization to handle uncertainty in data defining DMISOCPs by introducing two-stage stochastic mixed integer second-order cone programs (SMISCOPs) (with recourse). An application of class of problems will be described.*

Keywords: Stochastic programming; Mixed integer programming; Recourse; Second-order cone programming

1. Introduction

In deterministic mixed integer second-order cone programs (DMISCOPs) [6], a linear objective function is minimized over the intersection of an affine set and a product of second-order (Lorentz) cones, and an additional constraint that requires a subset of the variables attain integers values. We refer to them as *deterministic* mixed integer second-order cone programs since data defining them are deterministic. Deterministic 0-1 second-order cone programs (0-1DSCOPs) [6] are DMISCOPs but the variables that must take integer values are restricted to be binary.

In some applications we cannot specify the model entirely because it depends on information which is not available at the time of formulation but will only be determined at some point in the future. Stochastic programs have been studied since 1950âĂŹs to find optimal decisions in problems with uncertainty in data. See [5], [20], [4], [10], [13] and references contained therein. In particular, two-stage stochastic mixed integer linear programs (SMILPs) have been formulated to handle uncertainty in data defining mixed integer linear programs [16]. Some algorithm have been developed recently for solving SMILPs (see for example [15], [14]).

In this paper, we propose a new class of optimization problems to handle uncertainty in data defining DMISOCPs by introducing two-stage stochastic mixed integer second-order cone programs (SMISCOPs) (with recourse). We also describe an application of this new class of problems in stochastic mixed integer optimization,

1.1 Notations

We begin by introducing some notations that we use in the sequel. The notations in this part follows that of Alizadeh and Goldfarb [1] and Todd [18].

Let $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{n\vee n}$ denote the vector spaces of real $m\times n$ matrices and real symmetric $n\times n$ matrices respectively. For $U,V\in\mathbb{R}^{n\vee n}$, we write $U\succeq 0$ ($U\succ 0$) to mean that U is positive semidefinite (positive definite), and $U\succeq V$ or $V\preceq U$ to mean that $U-V\succeq 0$.

We use "," for adjoining vectors and matrices in a row, and use ";" for adjoining them in a column. So, for example, if $\boldsymbol{x},\boldsymbol{y}$, and $\boldsymbol{z}$ are vectors, the following are equivalent:

$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{z}\end{pmatrix}=(\boldsymbol{x}^{\mathsf T},\boldsymbol{y}^{\mathsf T},\boldsymbol{z}^{\mathsf T})^{\mathsf T}=(\boldsymbol{x};\boldsymbol{y};\boldsymbol{z}).$$

If $\mathcal{A}\subseteq\mathbb{R}^k$ and $\mathcal{B}\subseteq\mathbb{R}^l$, then the Cartesian product of $\mathcal{A}\times\mathcal{B}:=\{(\boldsymbol{x};\boldsymbol{y}):\boldsymbol{x}\in\mathcal{A}\text{ and }\boldsymbol{y}\in\mathcal{B}\}$.

For each vector $\boldsymbol{x}\in\mathbb{R}^k$ indexed from 0, we write $\bar{\boldsymbol{x}}$ for the sub-vector consisting of entries 1 through $k-1$; therefore $\boldsymbol{x}=(x_0;\bar{\boldsymbol{x}})$.

The second-order cone (also known as the quadratic, Lorentz, or the ice-cream cone) of dimension n is defined as $\mathcal{Q}_n:=\{\boldsymbol{x}=(x_0;\bar{\boldsymbol{x}})\in\mathbb{R}\times\mathbb{R}^{n-1}:x_0\geq||\bar{\boldsymbol{x}}||\}$ where $||\cdot||$ denotes the Euclidean norm. It is well known that the cone $\mathcal{Q}_2$ is convex, pointed, closed and with a nonempty interior.

We write $\boldsymbol{x}\succeq\boldsymbol{0}$ to mean that $\boldsymbol{x}\in\mathcal{Q}_n$, and $\boldsymbol{x}\succeq_{\langle n_1,n_2,\cdots,n_r\rangle}\boldsymbol{0}$ to mean that $\boldsymbol{x}\in\mathcal{Q}_{n_1}\times\mathcal{Q}_{n_2}\times\cdots\times\mathcal{Q}_{n_r}$. For simplicity, we write $\boldsymbol{x}\succeq_{\langle n_1,n_2,\cdots,n_r\rangle}\boldsymbol{0}$ as $\boldsymbol{x}\succeq_r\boldsymbol{0}$ when $n_1,n_2,\cdots,n_r$ are known from the context. We also write $\boldsymbol{x}\succeq_r\boldsymbol{y}$ or $\boldsymbol{y}\preceq_r\boldsymbol{x}$ to mean that $\boldsymbol{x}-\boldsymbol{y}\succeq_r\boldsymbol{0}$.

It is immediately seen that, for every vector $\boldsymbol{x}\in\mathbb{R}^n$ where $n=\sum_{i=1}^r n_i$, $\boldsymbol{x}\succeq_r\boldsymbol{0}$ if and only if $\boldsymbol{x}$ is partitioned conformally as $\boldsymbol{x}=(\boldsymbol{x_1};\boldsymbol{x_2};\cdots;\boldsymbol{x_r})$ and $\boldsymbol{x_i}\succeq\boldsymbol{0}$ for $i=1,2,\cdots,r$.

2. Definitions of SMISOCP with Recourse

An SMISOCP with recourse in primal standard form is defined based on deterministic data $A \in \mathbb{R}^{m_1 \times n_1}, \boldsymbol{b} \in \mathbb{R}^{m_1}$ and $\boldsymbol{c} \in \mathbb{R}^{n_1}$ and random data $T \in \mathbb{R}^{m_2 \times n_1}, W \in \mathbb{R}^{m_2 \times n_2}, \boldsymbol{h} \in \mathbb{R}^{m_2}$ and $\boldsymbol{d} \in \mathbb{R}^{n_2}$ whose realizations depend on an underlying outcome ω in an event space Ω with a known probability function $\mathbb{P}$. Given this data, a two-stage SMISOCP with recourse in *primal standard form* is

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}\,[Q(\boldsymbol{x},\omega)] \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq_{r_1} \boldsymbol{0} \\ & x_k \in [\alpha_k, \beta_k] \bigcap \mathbb{Z},\ k \in \Gamma \end{array} \tag{1}$$

where r_1 divides $n_1, \Gamma \subset \{1, 2, \cdots, n_1\}$, the first-stage decision variable $\boldsymbol{x} \in \mathbb{R}^{n_1}$ has some of its components x_k ($k \in \Gamma$) with integer values and bounded by $\alpha_k, \beta_k \in \mathbb{R}$, and $Q(\boldsymbol{x},\omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y} \\ \text{s.t.} & T(w)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & \boldsymbol{y} \succeq_{r_2} \boldsymbol{0} \\ & y_l \in [\gamma_l, \delta_l] \bigcap \mathbb{Z},\ l \in \Lambda \end{array} \tag{2}$$

where r_2 divides $n_2, \Lambda \subset \{1, 2, \cdots, n_2\}$, the second-stage decision variable $\boldsymbol{y} \in \mathbb{R}^{n_2}$ has some of its components y_l ($l \in \Lambda$) with integer values and bounded by $\gamma_l, \delta_l \in \mathbb{R}$, and

$$\mathbb{E}[Q(\boldsymbol{x},\omega)] := \int_{\Omega} Q(\boldsymbol{x},\omega) P(d\omega).$$

This class of optimization problems may be termed as *stochastic mixed integer second-order cone programs* (SMISOCPs) with recourse. If the integrality constraints in (1) and (2) are restricted to be binary, then we get the problem

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}\,[Q(\boldsymbol{x},\omega)] \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq_{r_1} \boldsymbol{0} \\ & x_k \in \{0, 1\},\ k \in \Gamma \end{array} \tag{3}$$

where r_1 divides $n_1, \Gamma \subset \{1, 2, \cdots, n_1\}$, the first-stage decision variable $\boldsymbol{x} \in \mathbb{R}^{n_1}$ has some of its components x_k ($k \in \Gamma$) with integer values and bounded by $\alpha_k, \beta_k \in \mathbb{R}$, and $Q(\boldsymbol{x},\omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y} \\ \text{s.t.} & T(w)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & \boldsymbol{y} \succeq_{r_2} \boldsymbol{0} \\ & y_l \in \{0, 1\},\ l \in \Lambda \end{array} \tag{4}$$

where r_2 divides $n_2, \Lambda \subset \{1, 2, \cdots, n_2\}$, the first-stage decision variable $\boldsymbol{y} \in \mathbb{R}^{n_2}$ has some of its components y_l ($l \in \Lambda$) with integer values and bounded by $\gamma_l, \delta_l \in \mathbb{R}$, and

$$\mathbb{E}[Q(\boldsymbol{x},\omega)] := \int_{\Omega} Q(\boldsymbol{x},\omega) P(d\omega).$$

This class of optimization problems may be termed as *stochastic 0-1 second-order cone programs* (0-1SSOCPs) with recourse.

3. Two general classes of problems can be cast as SMISOCPs

In this section we describe two general classes of problems that can be cast as MISSOCPs.

3.1 Stochastic mixed integer linear programs

If $r_1 = n_1$, then $x_i \in \mathcal{Q}_2^1 = \{t \in \mathbb{R} : t \geq 0\}$ for each $i = 1, 2, \cdots, n_1$. Thus the constraint $\boldsymbol{x} \succeq_{n_1} \boldsymbol{0}$ means the same as $\boldsymbol{x} \geq \boldsymbol{0}$, i.e., $\boldsymbol{x}$ lies in the nonnegative orthant of $\mathbb{R}^{n_1}$. Similarly, if $n_2 = r_2$ in (2), then $\boldsymbol{y}$ lies in the nonnegative orthant of $\mathbb{R}^{n_2}$. Thus, when both $n_1 = r_1$ in (1) and $n_2 = r_2$ in (2), then the SMISOCP problem (1, 2) reduces to the problem

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}\,[Q(\boldsymbol{x},\omega)] \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & x_k \in [\alpha_k, \beta_k] \bigcap \mathbb{Z},\ k \in \Gamma \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

where $\Gamma \subset \{1, 2, \cdots, n_1\}$, the first-stage decision variable $\boldsymbol{x} \in \mathbb{R}^{n_1}$ has some of its components x_k ($k \in \Gamma$) with integer values and bounded by $\alpha_k, \beta_k \in \mathbb{R}$, and $Q(\boldsymbol{x},\omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y} \\ \text{s.t.} & T(w)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & y_l \in [\gamma_l, \delta_l] \bigcap \mathbb{Z},\ l \in \Lambda \\ & \boldsymbol{y} \geq \boldsymbol{0} \end{array}$$

where $\Lambda \subset \{1, 2, \cdots, n_2\}$, the second-stage decision variable $\boldsymbol{y} \in \mathbb{R}^{n_2}$ has some of its components y_l ($l \in \Lambda$) with integer values and bounded by $\gamma_l, \delta_l \in \mathbb{R}$, and

$$\mathbb{E}[Q(\boldsymbol{x},\omega)] := \int_{\Omega} Q(\boldsymbol{x},\omega) P(d\omega).$$

Thus, SMILP problems can be cast as SMISOCP problems.

3.2 Stochastic mixed integer quadratic programs

Stochastic quadratic programs (SMIQPs) can also be cast as SMISOCPs. To demonstrate this, recall that a two-stage SMIQP (with recourse) is defined based on deterministic data $C \in \mathbb{R}^{n_1 \vee\ n_1}, C \succ 0, \boldsymbol{c} \in \mathbb{R}^{n_1}, A \in \mathbb{R}^{m_1 \times n_1}$ and $\boldsymbol{b} \in \mathbb{R}^{m_1}$; and random data $H \in \mathbb{R}^{n_2 \vee\ n_2}, H \succ 0, \boldsymbol{d} \in \mathbb{R}^{n_2}, T \in \mathbb{R}^{m_2 \times n_1}, W \in \mathbb{R}^{m_2 \times n_2}$, and $\boldsymbol{h} \in \mathbb{R}^{m_2}$ whose realizations depend on an underlying outcome in an event space Ω with a known probability function $\mathbb{P}$. Given this data, an SMIQP with recourse is

$$\begin{array}{ll} \min & q_1(\boldsymbol{x},\omega) = \boldsymbol{x}^{\mathsf{T}} C \boldsymbol{x} + \boldsymbol{c}^{\mathsf{T}} \boldsymbol{x} + \mathbb{E}[Q(\boldsymbol{x},\omega)] \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & x_k \in \{0,1\},\ k \in \Gamma \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{5}$$

where $\Gamma \subset \{1, 2, \cdots, n_1\}$, the first-stage decision variable $\boldsymbol{x} \in \mathbb{R}^{n_1}$ has some of its components x_k ($k \in \Gamma$) with integer values and bounded by $\alpha_k, \beta_k \in \mathbb{R}$, and $Q(\boldsymbol{x},\omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & q_2(\boldsymbol{y},\omega) = \boldsymbol{y}^{\mathsf{T}} H(\omega) \boldsymbol{y} + \boldsymbol{d}(\omega)^{\mathsf{T}} \boldsymbol{y} \\ \text{s.t.} & T(\omega)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & y_l \in [\gamma_l, \delta_l] \bigcap \mathbb{Z},\ l \in \Lambda \\ & \boldsymbol{y} \geq \boldsymbol{0} \end{array} \tag{6}$$

where $\Lambda \subset \{1, 2, \cdots, n_2\}$, the second-stage decision variable $\boldsymbol{y} \in \mathbb{R}^{n_2}$ has some of its components y_l ($l \in \Lambda$) with integer values and bounded by $\gamma_l, \delta_l \in \mathbb{R}$, and

$$\mathbb{E}[Q(\boldsymbol{x},\omega)] := \int_{\Omega} Q(\boldsymbol{x},\omega) P(d\omega).$$

Observe that the objective function of (5) can be written as (see [1])

$$q_1(\boldsymbol{x}_1,\omega) = ||\bar{\boldsymbol{u}}||^2 + \mathbb{E}[Q(\boldsymbol{x},\omega)] - \frac{1}{4} \boldsymbol{c}^{\mathsf{T}} C^{-1} \boldsymbol{c},$$

where

$$\bar{\boldsymbol{u}} = C^{{}^1/_2} \boldsymbol{x} + \frac{1}{2} C^{-{}^1/_2} \boldsymbol{c}.$$

Similarly, the objective function of (6) can be written as

$$q_2(\boldsymbol{y},\omega) = ||\bar{\boldsymbol{v}}||^2 - \frac{1}{4}\ \boldsymbol{d}(\omega)^{\mathsf{T}} H(\omega)^{-1} \boldsymbol{d}\ (\omega)$$

where

$$\bar{\boldsymbol{v}} = H(\omega)^{{}^1/_2} \boldsymbol{y} + \frac{1}{2} H(\omega)^{-{}^1/_2} \boldsymbol{d}(\omega).$$

Thus, problem (5, 6) can be transformed into the SMISOCP:

$$\begin{array}{ll} \min & u_0 \\ \text{s.t.} & \bar{\boldsymbol{u}} - C^{{}^1/_2} \boldsymbol{x} = \frac{1}{2}\ C^{-{}^1/_2} \boldsymbol{c} \\ & A\boldsymbol{x} = \boldsymbol{b} \\ & x_k \in [\alpha_k, \beta_k] \bigcap \mathbb{Z},\ k \in \Gamma \\ & (u_0; \bar{\boldsymbol{u}}) \succeq \boldsymbol{0} \\ & \boldsymbol{x} \geq 0 \end{array} \tag{7}$$

where $Q(\boldsymbol{x},\omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & v_0 \\ \text{s.t.} & \bar{\boldsymbol{v}} - H(\omega)^{{}^1/_2} \boldsymbol{y} = \frac{1}{2}\ H(\omega)^{-{}^1/_2} \boldsymbol{d}(\omega) \\ & T(\omega)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & y_l \in [\gamma_l, \delta_l] \bigcap \mathbb{Z},\ l \in \Lambda \\ & (u_0; \bar{\boldsymbol{v}}) \succeq \boldsymbol{0} \\ & \boldsymbol{y} \geq \boldsymbol{0} \end{array} \tag{8}$$

where

$$\mathbb{E}[Q(\boldsymbol{x},\omega)] := \int_{\Omega} Q(\boldsymbol{x},\omega) P(d\omega).$$

Note that the SMIQP problem (5, 6) and the SMISOCP problem (7, 8) will have the same minimization, but their optimal objective values are equal up to constants. More precisely, the difference between the optimal objective values of (6, 8) would be $-\frac{1}{2}\boldsymbol{d}(\omega)^{\mathsf{T}} H(\omega)^{-1}\ \boldsymbol{d}(\omega)$. Similarly, the optimal objective values of (5, 6) and (7, 8) will differ by

$$-\frac{1}{2}\ \boldsymbol{c}^{\mathsf{T}} C^{-1} \boldsymbol{c} - \frac{1}{2} \int_{\Omega} \left(\boldsymbol{d}(\omega)^{\mathsf{T}}\ H(\omega)^{-1} \boldsymbol{d}(\omega)\right) P(d\omega).$$

It is interesting to note that we can use the transformation described in this part to formulate an SMISOCP model for capital budgeting problems with a mean-variance objective described in [2]. In [2] the authors ignored the financing structure and considered a simple assumption that all given projects have a *fixed* available budget, and then, in order to fit their approach for deriving cutting planes, they transformed the problem from its model in 0-1DQP into a 0-1DSOCP model. But we believe that it is much closer to the reality to assume that we have a *random* budget for the projects. Consequently, it is more convenient to consider the stochastic version of this problem and hence to transform it from the resulting 0-1SQP model into a 0-1SSOCP model.

4. An application: Stochastic discrete Euclidean facility location problems

In *facility location problems* (FLPs) we are interested in choosing a location to build a new facility or locations to build multiple new facilities so that an appropriate measure of distance from the new facilities to existing facilities is minimized. FLPs arise locating airports, regional campuses, wireless communications towers, etc. The following are some ways of classifying FLPs (see also [17]):

- We can classify FLPs based on the number of new facilities in the following sense: if we add only one new facility then we get a problem known as a *single facility location problem* (SFLP), while if we add multiple new facilities instead of adding only one, then we get more a general problem known as a *multiple facility location problem* (MFLP).
- Another way of classification is based on the distance measure used in the model between the facilities. If we use the Euclidean distance then these problems are called *Euclidean facility location problems* (EFLPs), if we use the rectilinear distance then these problems are called *rectilinear facility location problems* (RFLPs).
- When the new facilities can be placed any place in solution space, the problem is called a *continuous facility location problem* (CFLP), but usually the investor needs the new facilities to be placed within *specific locations* (called nodes) and not in any place in the solution space. In this case the problem is called a *discrete facility location problem* (DFLP).
- In some applications, the locations of existing facilities cannot be fully specified because the locations of some

of them depend on information not available at the time when decision needs to be made but will only be available at a later point in time. In this case, we are interested in *stochastic facility location problems*. When the locations of all old facilities are fully specified, FLPs are called *deterministic facility location problems*.

FLPs have seen a great deal of recent research activity. For further details, consult the book of Tompkins and *et al.* [17]. In particular, deterministic Euclidean facility location problems are often cited as an application of deterministic second-order cone programs (see for example [19] and [11]). In this subsection, we consider (both single and multiple) stochastic discrete Euclidean facility location problems when, in particular, some of the variables are restricted to be integer variables.

4.1 Stochastic discrete Euclidean single facility location problem

In deterministic Euclidean single facility location problems, we are interested in choosing a location to build a new facility among existing facilities so that this location minimizes the sum of a weighted distance to all existing facilities.

Assume that we are given r existing facilities represented by the fixed points $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_r$ in $\mathbb{R}^n$, and we plan to place a new facility represented by $\boldsymbol{x}$ so that we minimize the weighted sum of the distances between $\boldsymbol{x}$ and each of the points $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_r$. This leads us to the problem

$$\min \quad \sum_{i=1}^{r} w_i \, ||\boldsymbol{x} - \boldsymbol{a}_i||$$

or, alternatively, to the problem

$$\begin{array}{ll} \min & \sum_{i=1}^{r} w_i \, t_i \\ \text{s.t.} & (t_1; \boldsymbol{x} - \boldsymbol{a}_1; \cdots; t_r; \boldsymbol{x} - \boldsymbol{a}_r) \succeq_r \mathbf{0} \end{array}$$

where w_i is the weight associated with the ith existing facility and the new facility for $i = 1, 2, \ldots, r$.

Before we describe the stochastic version of this generic application, we indicate a more concrete version of it. Assume that we have a new city with many suburbs and we want to build a hospital for treating the residents of this city. Some people live in the city at the present time. As the city expands, many houses in new suburbs need to be built and the locations of these suburbs will be known in the future in different sides of the city. Our goal is to find the best location of this hospital so that it can serve the current suburbs and the new ones. This location must be determined at the current time and before information about the locations of the new suburbs become available.

Generally speaking, let $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{r_1}$ be fixed points in $\mathbb{R}^n$ representing the coordinates of r_1 existing fixed facilities and $\tilde{\boldsymbol{a}}_1(\omega), \tilde{\boldsymbol{a}}_2(\omega), \cdots, \tilde{\boldsymbol{a}}_{r_2}(\omega)$ be random points in $\mathbb{R}^n$ representing the coordinates of r_2 random facilities who realizations depends on an underlying outcome ω in an event space Ω with a known probability function $\mathbb{P}$.

Suppose that at present we do not know the realizations of r_2 random facilities, and that at some point in time in future the realizations of these r_2 random facilities become known.

Our goal is to locate a new facility $\boldsymbol{x}$ that minimizes the weighted sums of the distance between the new facility and each one of the existing fixed facilities and also minimizes the expected weighted sums of the distance between the new facility and the realization of each one of the random facilities. Note that this decision needs to be made before the realizations of the r_2 random facilities become available. We consider the discrete version of the problem by assuming that the new facility needs to be placed within specific locations and not in any place in 2- or 3- (or higher) dimensional space. Let the points $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k \in \mathbb{R}^n$ represent these specific locations. So, we add the constraint $\boldsymbol{x} \in \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k\}$. Clearly, the above constraint can be replaced by the following linear and binary constraints:

$$\begin{array}{l} \boldsymbol{x} = \boldsymbol{v}_1 \, y_1 + \boldsymbol{v}_2 \, y_2 + \cdots + \boldsymbol{v}_k \, y_k, \\ y_1 + y_2 + \cdots + y_k = 1, \text{ and} \\ (y_1, y_2, \cdots, y_k) \in \{0, 1\}^k. \end{array}$$

This leads us to the following 0-1SSOCP model:

$$\begin{array}{ll} \min & \sum_{i=1}^{r_1} w_i \, t_i + \mathbb{E}\,[Q(\boldsymbol{x}; \boldsymbol{y}, \omega)] \\ \text{s.t.} & (t_1; \boldsymbol{x} - \boldsymbol{a}_1; \cdots; t_{r_1}; \boldsymbol{x} - \boldsymbol{a}_{r_1}) \succeq_{r_1} \mathbf{0} \\ & \boldsymbol{x} = \boldsymbol{v}_1 \, y_1 + \boldsymbol{v}_2 \, y_2 + \cdots + \boldsymbol{v}_k \, y_k \\ & \mathbf{1}^\mathsf{T} \, \boldsymbol{y} = 1, \ \boldsymbol{y} \in \{0, 1\}^k \end{array}$$

where $Q(\boldsymbol{x}; \boldsymbol{y}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \sum_{j=1}^{r_2} \tilde{w}_j(\omega) \, \tilde{t}_j \\ \text{s.t.} & (\tilde{t}_1; \boldsymbol{x} - \tilde{\boldsymbol{a}}_1(\omega); \cdots; \tilde{t}_{r_2}; \boldsymbol{x} - \tilde{\boldsymbol{a}}_{r_2}(\omega)) \succeq_{r_2} \mathbf{0} \\ & \boldsymbol{x} = \boldsymbol{v}_1 \, y_1 + \boldsymbol{v}_2 \, y_2 + \cdots + \boldsymbol{v}_k \, y_k \\ & \mathbf{1}^\mathsf{T} \, \boldsymbol{y} = 1, \ \boldsymbol{y} \in \{0, 1\}^k \end{array}$$

and

$$\mathbb{E}[Q(\boldsymbol{x}; \boldsymbol{y}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}; \boldsymbol{y}, \omega) P(d\omega).$$

where w_i is the weight associated with the ith existing facility and the new facility for $i = 1, 2, \ldots, r_1$ and $\tilde{w}_j(\omega)$ is the weight associated with the jth random existing facility and the new facility for $j = 1, 2, \ldots, r_2$.

Sometimes we may need the specific points have to attain integer values. In most cities of the world that were planned, streets are laid out on a grid plan, so that city is subdivided into small numbered blocks that are square or rectangular. Figure 2 shows the blocks of Chicago in 1857. In this case, usually the investor needs the new facility to be placed on of the corners of the city blocks. Thus, let us assume that the variable $\boldsymbol{x}$ lies in the hyperrectangle $\Xi^n := \{\boldsymbol{x} : \boldsymbol{\zeta} \leq \boldsymbol{x} \leq \boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbb{R}^n, \boldsymbol{\eta} \in \mathbb{R}^n\}$ and has to attain specific points

Fig. 1: The regular pattern of square or rectangular city blocks is very common among American cities. This map shows the blocks of Chicago in 1857.

in the grid $\Xi^n \bigcap \mathbb{Z}^n$. Then we simply solve the following SMISOCP model:

$$\begin{array}{ll} \min & \sum_{i=1}^{r_1} w_i\ t_i + \mathbb{E}\ [Q(\boldsymbol{x}, \omega)] \\ \text{s.t.} & (t_1; \boldsymbol{x} - \boldsymbol{a}_1; \cdots; t_{r_1}; \boldsymbol{x} - \boldsymbol{a}_{r_1}) \succeq_{r_1} \boldsymbol{0} \\ & \boldsymbol{x} \in \Xi^n \bigcap \mathbb{Z}^n \end{array}$$

where $Q(\boldsymbol{x}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \sum_{j=1}^{r_2} \tilde{w}_j(\omega)\ \tilde{t}_j \\ \text{s.t.} & (\tilde{t}_1; \boldsymbol{x} - \tilde{\boldsymbol{a}}_1(\omega); \cdots; \tilde{t}_{r_2}; \boldsymbol{x} - \tilde{\boldsymbol{a}}_{r_2}(\omega)) \succeq_{r_2} \boldsymbol{0} \\ & \boldsymbol{x} \in \Xi^n \bigcap \mathbb{Z}^n \end{array}$$

and

$$\mathbb{E}[Q(\boldsymbol{x}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}, \omega) P(d\omega).$$

4.2 Stochastic discrete Euclidean multiple facility location problem

If we consider the concrete model described in §4, and suppose that we want to build three hospitals for this city, or build a hospital, a university, and a fire station then we get a multiple facility version of the model. Generally, in order to be precise only the latest information of the random facilities is used. This may require an increasing or decreasing of the number of the new facilities after the latest information about the random facilities become available. For simplicity, let us assume that the number of new facilities was previously known and fixed and we add m new facilities, namely $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m \in \mathbb{R}^n$, instead of adding only one. We also assume that the variables $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m$ need to be placed within specific locations represented by the points $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k \in \mathbb{R}^n$.

We have two cases depending whether or not there is an interaction among the new facilities in the underlying model. If there is no interaction between the new facilities, we are just concerned in minimizing the weighted sums of the distance between each one of the new facilities on one hand and each one of the fixed facilities and the realization of each one of the random facilities. In other words, we solve the following 0-1SSOCP model:

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r_1} w_{ij}\ t_{ij} + \mathbb{E}\ [Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m; \boldsymbol{y}, \omega)] \\ \text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{r_1 j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1}) \succeq_{r_1} \boldsymbol{0} \\ & \text{where}\ \ j = 1, 2, \cdots, m \\ & \boldsymbol{x}_j = \boldsymbol{v}_1\ y_1 + \boldsymbol{v}_2\ y_2 + \cdots + \boldsymbol{v}_k\ y_k \\ & \text{where}\ \ j = 1, 2, \cdots, m \\ & \boldsymbol{1}^{\mathsf{T}}\ \boldsymbol{y} = 1,\ \boldsymbol{y} \in \{0, 1\}^k, \end{array}$$

where $Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m; \boldsymbol{y}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r_2} \tilde{w}_{ij}(\omega)\ \tilde{t}_{ij} \\ \text{s.t.} & (\tilde{t}_{1j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_1(\omega); \cdots; \tilde{t}_{r_2 j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{r_2}(\omega)) \succeq_{r_2} \boldsymbol{0} \\ & \text{where}\ \ j = 1, 2, \cdots, m \\ & \boldsymbol{x}_j = \boldsymbol{v}_1\ y_1 + \boldsymbol{v}_2\ y_2 + \cdots + \boldsymbol{v}_k\ y_k \\ & \text{where}\ \ j = 1, 2, \cdots, m \\ & \boldsymbol{1}^{\mathsf{T}}\ \boldsymbol{y} = 1,\ \boldsymbol{y} \in \{0, 1\}^k, \end{array}$$

and

$$\mathbb{E}[Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m; \boldsymbol{y}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m; \boldsymbol{y}, \omega) P(d\omega).$$

where w_{ij} is the weight associated with the ith existing facility and the jth new facility for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, r_1$, and $\tilde{w}_{ij}(\omega)$ is the weight associated with the ith random existing facility and the jth new facility for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, r_2$.

If interaction exists among the new facilities, then, in addition to the above requirements, we need to minimize the sum of the Euclidean distances between each pair of the new facilities. In this case, we are interested in a model of the form:

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r_1} w_{ij} t_{ij} + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\ & + \mathbb{E}\ [Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m; \boldsymbol{y}, \omega)] \\ \text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{r_1 j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1}) \succeq_{r_1} \boldsymbol{0} \\ & \text{where}\ \ j = 1, 2, \cdots, m \\ & (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)} \boldsymbol{0} \\ & \text{where}\ \ j = 1, 2, \cdots, m-1 \\ & \boldsymbol{x}_j = \boldsymbol{v}_1\ y_1 + \boldsymbol{v}_2\ y_2 + \cdots + \boldsymbol{v}_k\ y_k \\ & \text{where}\ \ j = 1, 2, \cdots, m \\ & \boldsymbol{1}^{\mathsf{T}}\ \boldsymbol{y} = 1,\ \boldsymbol{y} \in \{0, 1\}^k, \end{array}$$

where $Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m; \boldsymbol{y}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r_2} \tilde{w}_{ij}(\omega)\, \tilde{t}_{ij} + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\ \text{s.t.} & (\tilde{t}_{1j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_1(\omega); \cdots; \tilde{t}_{r_2 j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{r_2}(\omega)) \succeq_{r_2} \mathbf{0} \\ & \text{where } j = 1, 2, \cdots, m \\ & (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)} \mathbf{0} \\ & \text{where } j = 1, 2, \cdots, m-1 \\ & \boldsymbol{x}_j = \boldsymbol{v}_1\, y_1 + \boldsymbol{v}_2\, y_2 + \cdots + \boldsymbol{v}_k\, y_k \\ & \text{where } j = 1, 2, \cdots, m \\ & \mathbf{1}^{\mathsf{T}} \boldsymbol{y} = 1,\ \boldsymbol{y} \in \{0, 1\}^k, \end{array}$$

and

$$\mathbb{E}[Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m; \boldsymbol{y}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m; \boldsymbol{y}, \omega) P(d\omega).$$

where $\hat{w}_{jj'}$ is the weight associated with the new facilities j' and j for $j' = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, m$.

If we need some specific points have to attain integer values, then for each $k \in \Delta \subset \{1, 2, \cdots, m\}$, we assume that the variable $\boldsymbol{x}_k$ lies in the hyperrectangle $\Xi_k^n \equiv \{\boldsymbol{x}_k : \boldsymbol{\zeta}_k \leq \boldsymbol{x}_k \leq \boldsymbol{\eta}_k, \boldsymbol{\zeta}_k \in \mathbb{R}^n, \boldsymbol{\eta}_k \in \mathbb{R}^n\}$ and has to be integer-valued, i.e. $\boldsymbol{x}_k$ must be in the grid $\Xi_k^n \bigcap \mathbb{Z}^n$.

Thus, if there is no interaction between the new facilities, we solve the following SMISOCP model:

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r_1} w_{ij}\, t_{ij} + \mathbb{E}\, [Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_{r_1}, \omega)] \\ \text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{r_1 j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1}) \succeq_{r_1} \mathbf{0} \\ & \text{where } j = 1, 2, \cdots, m \\ & \boldsymbol{x}_k \in \Xi_k^n \bigcap \mathbb{Z}^n,\ k \in \Delta, \end{array}$$

where $Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r_2} \tilde{w}_{ij}(\omega)\, \tilde{t}_{ij} \\ \text{s.t.} & (\tilde{t}_{1j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_1(\omega); \cdots; \tilde{t}_{r_2 j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{r_2}(\omega)) \succeq_{r_2} \mathbf{0} \\ & \text{where } j = 1, 2, \cdots, m \\ & \boldsymbol{x}_k \in \Xi_k^n \bigcap \mathbb{Z}^n,\ k \in \Delta, \end{array}$$

and

$$\mathbb{E}[Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)] := \int_{\Omega} Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega) P(d\omega).$$

If interaction exists among the new facilities, then we are interested in a model of the form:

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r_1} w_{ij} t_{ij} + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\ & + \mathbb{E}\, [Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_{r_1}, \omega)] \\ \text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{r_1 j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1}) \succeq_{r_1} \mathbf{0} \\ & \text{where } j = 1, 2, \cdots, m \\ & (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)} \mathbf{0} \\ & \text{where } j = 1, 2, \cdots, m-1 \\ & \boldsymbol{x}_k \in \Xi_k^n \bigcap \mathbb{Z}^n,\ k \in \Delta, \end{array}$$

where $Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r_2} \tilde{w}_{ij}(\omega)\, \tilde{t}_{ij} \\ & + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\ \text{s.t.} & (\tilde{t}_{1j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_1(\omega); \cdots; \tilde{t}_{r_2 j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{r_2}(\omega)) \succeq_{r_2} \mathbf{0} \\ & \text{where } j = 1, 2, \cdots, m \\ & (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)} \mathbf{0} \\ & \text{where } j = 1, 2, \cdots, m-1 \\ & \boldsymbol{x}_k \in \Xi_k^n \bigcap \mathbb{Z}^n,\ k \in \Delta, \end{array}$$

and

$$\mathbb{E}[Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)] := \int_{\Omega} Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega) P(d\omega).$$

5. Future research directions

In this paper we introduced a new class of problems for stochastic mixed integer programming that may be referred as stochastic mixed integer second-order cone programs with recourse. Stochastic mixed integer second-order cone programs generalize both stochastic mixed integer linear programs and stochastic mixed integer quadratic programs. Our development is indeed significant in value, because it gives us a new methodology to cover those applications that cannot be captured by stochastic mixed integer linear and quadratic programs. In terms of modeling, beyond the application described in §4, it would be interesting to investigate other applications of this new class of optimization problems. For example, in [8] Fampa and Maculan proposed a deterministic mixed integer second-order cone programming formulation of the Euclidean Steiner tree problem (in which the set of nodes in the connection is *fixed* over time). Based on this formulation, we can describe a stochastic mixed integer second-order cone programming formulation of a related problem called *dynamic* Euclidean Steiner tree problem (where the set of nodes in the connection changes over time) proposed by Imase and Waxman in [9] and motivated by multipoint routing in communication networks.

It is useful to develop algorithm for SMISOCPs. A forthcoming paper will focus on developing a decomposition-based branch-and-bound algorithm for solving this new class of problems by extending the work of Sherali and Zhu [15] (see also [14]).

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